Infinitesimal Conformal Deformations of Triangulated Surfaces

Master Thesis

by

Wai Yeung Lam

TECHNISCHE UNIVERSITÄT BERLIN
Fakultät II - Institut für Mathematik

Prof. Ulrich Pinkall

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Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig und eigenhändig sowie ohne unerlaubte fremde Hilfe und ausschließlich unter Verwendung der aufgeführten Quellen und Hilfsmittel angefertigt habe.

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Introduction

Immersions of a surface into a space of constant curvature are traditionally studied in differential geometry. Some classical problems look for immersions with special mean curvature. The Bonnet problems are concerned with the existence and uniqueness of immersed surfaces in $\mathbb{R}^3$ with a prescribed metric and mean curvature. Higher genus surfaces with constant mean curvature are also being studied. The characterization of Willmore surfaces, which are the critical points of the Willmore functional $\int_M H^2|df|^2$, is also under investigation. These problems involve control over mean curvatures H and are still not completely solved (Abresch et al., 2013).

Quaternionic analysis provides a coordinate-free way to handle these classical problems (Kamberov et al., 1998; Pedit and Pinkall, 1998). Given a Riemann surface and a conformal immersion f into $\mathbb{R}^3$, the space of conformal immersions regularly homotopic to f is generically parameterized by the mean curvature half-density $H|df|$. For a Riemann sphere, these mean curvature half-densities form a hyper-surface in the vector space of all half-densities. Instead of point positions, this method works with geometric quantities directly. It thus has better control over the mean curvature of conformal immersions to tackle the classical problems mentioned earlier.

This thesis considers a discrete analogue of the quaternion analysis. The relation between infinitesimal conformal deformations of triangulated surfaces in Euclidean space and their extrinsic geometry is shown. It is in the spirit of discrete differential geometry (Bobenko and Suris, 2008), with the aim to look for mathematical structures on triangulated surfaces as rich as in the smooth theory.

Looking for a discrete analogue to the smooth theory is natural. On one hand, from the results in the smooth theory, one wonders if there are similar techniques and structures on triangulated surfaces. On the other hand, a discrete theory may give a hint to phenomena on smooth surfaces by looking at analogous conditions.

There were related works in various directions. Numerical algorithms are achieved by discretizing equations from the smooth theory at the cost of losing mathematical structures. Gu and Yau (2003) achieved conformal parametrization of triangulated surfaces by computing a basis of discrete harmonic functions. Crane et al. (2011) considered conformal deformations of triangulated surfaces with extrinsic geometry, by discretizing equations from the quaternionic analysis. These methods lack the notion of conformal equivalence of triangulated surfaces.

Sensible discrete analogues of conformality were defined and found with nice property as in the smooth theory, which is in the spirit of discrete differential geometry. Thurston proposed circle packing, by defining circles on vertices of a triangulated mesh such that neighboring circles tangent to each other (Stephenson, 2005). It was generalized to circle pattern by enabling the circles intersecting. Two triangular meshes are conformal equivalent if the induced intersection angles of neighboring circles on edges are equal. Luo (2004) and Springborn et al. (2008) considered the conformal equivalence of two discrete metric by defining conformal factors on vertices. In particular, their relations to intrinsic geometry were studied.

The two respective methods found that discrete metrics within a conformal class are determined by the discrete curvatures on vertices.

Triangulated surfaces in Euclidean space with their extrinsic geometry were also studied. Bobenko and Schröder (2005) proposed discrete Willmore flow of triangulated surfaces by considering the intersection angles of neighboring circumcircles in space, which is Möbius invariant. And hence the Willmore functional defined is also Möbius invariance, as in the case of the smooth theory.

In this thesis, the approach to relate infinitesimal conformal deformations of triangulated surfaces with their extrinsic geometry is based on two ingredients. One is the notion of conformal equivalence of triangular meshes by defining conformal factors on vertices (Bobenko et al., 2010; Luo, 2004). In this way, any discrete metric on edges associates a conformal class to the triangulated surface. It is a better analogue to the smooth theory compared to circle packing (Stephenson, 2005) and circle pattern. In the smooth theory, any immersion of a surface into Euclidean space induces a conformal structure. However, for triangulated surfaces in Euclidean space, not every metric is a circle packing metric and circle pattern could not be naturally induced.

Another ingredient is a dual way to describe the local closedness condition under infinitesimal deformations, i.e., by prescribing the change of each triangular face and then imposing the closedness conditions on edges. This differs from the usual way, where changes to edge vectors are first prescribed and then the closedness conditions on faces are imposed. Figure 0.1 and Figure 0.2 illustrate how the local closedness conditions fail under the two descriptions. Indeed, most formulas analogous to the smooth theory appear in the dual description.

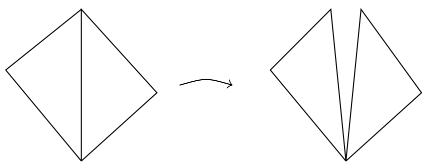

FIGURE 0.1. Change of faces are prescribed and a common edge splits

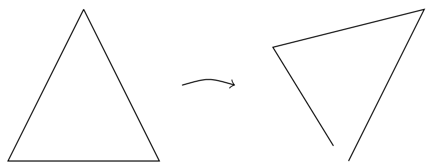

FIGURE 0.2. Change of edges are prescribed and a face does not close up

The following statement is a good guideline for the development of this thesis. It describes the tangent space of the space of all conformal immersions. It holds both in the sense of the smooth theory (Theorem 2.19) and our discrete theory (Theorem 3.29).

Claim 1. *Suppose $f : M \to \mathbb{R}^3$ is an immersion of a genus g-surface with the kernel of Dirac operator* $\operatorname{Ker} \mathrm{D}$ *of dimension* 4*. Then there exists an infinitesimal conformal deformation of f with a prescribed change of mean curvature half-density $\dot{\rho}\,|df|$ if and only if $\int_M \dot{\rho}\,|df|^2 = 0$ and $\dot{\rho}$ is L^2-perpendicular to $6g$ functions depending on the immersion f.*

In the smooth theory, the notations in the statement are clear, as studied in (Pedit and Pinkall, 1998; Richter, 1997). The details are reviewed in Chapter 2. However, their meaning in the discrete theory is not clear. Therefore, in Chapter 3 we discuss the discrete analogues of the above notations and their connection to the classical results. Namely, under infinitesimal conformal deformations, we have Table 1.

This thesis is divided into 4 chapters. Chapter 1 contains background knowledge covering quaternionic linear algebra and discrete differential forms. Hodge

Smooth Surfaces	Triangulated Surfaces
$\dot{\rho}\|df\|^2 = (H_t \dot{\|df_t\|})\|df\|$	$\dot{\rho}_i = \sum_{ij \in E:i} \frac{\dot{\alpha}_{ij}}{2} \|df(e_{ij})\|$
where H_t is the mean curvature of f_t	where α_{ij} is the dihedral angle on the edge e_{ij}
$\int_M \dot{\rho}\|df\|^2 = \int_M (H\dot{\|df\|}) = 0$	$\sum_{i \in V} \dot{\rho}_i = \sum_{ij \in E} \dot{\alpha}_{ij}\|df(e_{ij})\| = 0$ (Schläfli Formula)

Table 1. Some analogues between the smooth and our discrete theory

Decomposition Theorem for discrete differential forms would be derived. This chapter ends with a one-dimensional illustration of infinitesimal conformal deformations of surfaces. We would compare the space of plane curves with fixed length parametrized by curvature functions in both smooth and discrete cases.

Chapter 2 considers conformal deformations of smooth surfaces via quaternionic analysis. We first review their basic results and then focus on infinitesimal conformal deformations. Constraints of infinitesimal conformal deformations for High genus surfaces are derived. Most theorems here have a discrete analogue in the following chapter.

Chapter 3 is the main part of the thesis and focuses on infinitesimal conformal deformations of triangulated surfaces. We first review the definition conformal equivalence of triangulated surfaces and its nice property compared to the smooth theory. Then we derive the discrete Dirac operator by considering infinitesimal conformal deformations of triangulated surfaces. Constraints of infinitesimal conformal deformations for High genus surfaces are shown to correspond to the smooth theory nicely. It ends with the derivation of discrete Laplace operator—the cotangent Laplace formula.

Chapter 4 gives explicit examples for comparison. The choice of conformal equivalence and discrete notions derived in Chapter 3 is justified by comparing deformations of smooth and triangulated surfaces. The dimension of the kernel dim(Ker D) of the discrete Dirac operator is also calculated for several triangulated surfaces. We present examples with dim(Ker D) = 4 and dim(Ker D) > 4. They are motivated from the study of the rigidity of polyhedral surfaces.

Most of the results in this thesis are drawn or inspired from the blog "Discrete Spin Geometry", contributed by the discussions among Keenan Crane, Ulrich Pinkall, Peter Schroeder and Boris Springborn, in which I learned a lot from their insightful discussions.

CHAPTER 1

Background

This chapter establishes basic tools to prepare for the following chapters and illustrates the idea of parameterizing immersions by extrinsic geometric quantities. Section 1 and 2 would review notions of quaternions and discrete differential forms. Section 3 studies deformations of planar curves with a fixed length by curvature functions. Both smooth and discrete curves are considered. It serves as a 1-dimensional analogue of conformal deformations of surfaces.

1. Quaternionic Theory

Since we are going to study surfaces immersed in Euclidean space $\mathbb{R}^3$ and identify the Euclidean space as the imaginary part of the space of quaternions $\mathrm{Im}(\mathbb{H})$, we review some notations in the theory of quaternions.

Definition 1.1. *The space of quaternions $\mathbb{H}$ is a 4-dimensional vector space over $\mathbb{R}$ spanned by $1, i, j, k$ with multiplicative relations*

$$i^2 = j^2 = k^2 = -1,$$
$$ijk = -1.$$

One can check that $\mathbb{H}$ is a non-commutative field. In the following, we define notations similar to those in complex numbers.

Definition 1.2. *The real part and the imaginary part of quaternions are defined via*

$$\mathrm{Re}(\mathbb{H}) := \mathrm{Span}_{\mathbb{R}}(1),$$
$$\mathrm{Im}(\mathbb{H}) := \mathrm{Span}_{\mathbb{R}}(i, j, k) \cong \mathbb{R}^3.$$

Definition 1.3. *A conjugation is a linear operator on $\mathbb{H}$ such that for any $q = a + b\,i + c\,j + d\,k \in \mathbb{H}$ where $a, b, c, d \in \mathbb{R}$, the conjugation of q is*

$$\overline{q} = a - b\,i - c\,j - d\,k.$$

For any two quaternions $\lambda, \nu \in \mathbb{H}$, we have a formula for their product

$$\begin{aligned}\lambda\nu = &\mathrm{Re}(\lambda)\,\mathrm{Re}(\nu) - \langle \mathrm{Im}(\lambda), \mathrm{Im}(\nu)\rangle_{\mathbb{R}^3} \\ &+ \mathrm{Re}(\lambda)\,\mathrm{Im}(\nu) + \mathrm{Re}(\nu)\,\mathrm{Im}(\lambda) + \mathrm{Im}(\lambda) \times \mathrm{Im}(\nu)\end{aligned}$$

where $\langle\ ,\ \rangle_{\mathbb{R}^3}$ and $\times$ are the inner product and the cross product in $\mathbb{R}^3$. From this, we have

$$\overline{\lambda\nu} = \overline{\nu}\,\overline{\lambda},$$
$$\mathrm{Re}(\lambda\nu) = \mathrm{Re}(\nu\lambda).$$

Definition 1.4. *An inner product $\langle\ ,\ \rangle$ on $\mathbb{H}$ is defined such that for any two quaternions $\lambda, \nu \in \mathbb{H}$,*

$$\langle \lambda, \nu\rangle_{\mathbb{H}} := \mathrm{Re}(\lambda\overline{\nu})$$

and the norm $|\ |$ is defined by

$$|\lambda|^2 = \langle \lambda, \lambda\rangle_{\mathbb{H}}.$$

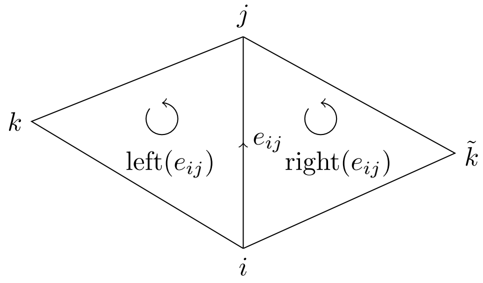

FIGURE 1.1. Orientation induced on triangles

From the product formula, the properties of an inner product can be verified easily. We then look at how a quaternion acts on $\mathbb{R}^3$ as a composition of scaling and rotation.

Lemma 1.5. *Let $\lambda \in \mathbb{H}$ be arbitrary. We write $\lambda = |\lambda|(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}T)$ for some $\alpha \in \mathbb{R}$ and a unit vector $T \in S^2 \subset \mathrm{Im}(\mathbb{H})$. Then, for any vector $W \in \mathrm{Im}(\mathbb{H})$,*

$$\overline{\lambda}W\lambda = |\lambda|^2\big((W - W^\perp) + \cos\alpha\, W^\perp + \sin\alpha\, T \times W^\perp\big)$$

where $W^\perp$ is the component of W perpendicular to T.

PROOF. We compute directly using the algebraic relations of quaternions. Then

$$\begin{aligned}
&\overline{\lambda}W\lambda \\
=&|\lambda|^2(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}T)W(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}T) \\
=&|\lambda|^2\big(W(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}T)(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}T) + 2\sin\frac{\alpha}{2}\,T \times W(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}T)\big) \\
=&|\lambda|^2(W + 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}T \times W - 2\sin^2\frac{\alpha}{2}W^\perp) \\
=&|\lambda|^2(W + \sin\alpha T \times W - (1-\cos\alpha)W^\perp) \\
=&|\lambda|^2(W - W^\perp + \cos\alpha W^\perp + \sin\alpha T \times W^\perp).
\end{aligned}$$

□

2. Discrete Differential Forms

The discrete surfaces studied in this thesis are closed oriented triangulated surfaces. We restrict the review of discrete differential forms to these surfaces in order to simplify the introduction. General introduction can be found in Desbrun et al. (2006). This section ends with the Hodge decomposition theorem of discrete differential forms.

Definition 1.6. *A triangulated surface is a connected topological surface, which is also a simplicial complex. The set of vertices (0-cells), edges (1-cells) and triangles (2-cells) are denoted as V, E and F.*

We denote $\tilde{E}$ as the set of all oriented edges. If $e \in \tilde{E}$, then $e \neq -e \in \tilde{E}$. In addition, for an oriented surface, its orientation induces an orientation on every triangle naturally. And we can talk about the left face and the right face of an oriented edges (Figure 1.1). In the following, all surfaces are assumed to be oriented and its orientation is fixed.

Definition 1.7. *A discrete function (0-form) f on a discrete surface M is a map $f : V \to \mathbb{R}$. The space of discrete functions on M is denoted as $\Omega^0(M)$.*

A discrete 1-form ω on a discrete surface M is a map $\omega : \tilde{E} \to \mathbb{R}$ such that for any edge $e \in \tilde{E}$, $\omega(-e) = -\omega(e)$. The space of all discrete 1-forms is written as $\Omega^1(M)$.

A discrete 2-form σ on a discrete surface M is a map $\sigma : F \to \mathbb{R}$. The space of all discrete 2-forms is written as $\Omega^2(M)$.

Definition 1.8. *The derivative of a discrete function f is a 1-form $d_0 f : \tilde{E} \to \mathbb{R}$ such that for any oriented edges $e \in \tilde{E}$,*

$$d_0 f(e_{ij}) = f(v_j) - f(v_i)$$

where the edge e_{ij} is from vertex v_i to v_j. And so the derivative is a map $d_0 : \Omega^0 \to \Omega^1$.

Definition 1.9. *The derivative of a 1-form ω is 2-form $d_1\omega : F \to \mathbb{R}$ such that for any oriented face $\phi_{ijk} \in F$,*

$$d_1\omega(\phi_{ijk}) = \omega(e_{ij}) + \omega(e_{jk}) + \omega(e_{ki})$$

where the order of (ijk) is compatible with the orientation of the triangle ϕ_{ijk}. (Fig. 1.1) And the derivative is a map $d_1 : \Omega^1 \to \Omega^2$.

Subscripts of the derivatives are omitted whenever the input is clear.

Definition 1.10. *We define the closedness of a discrete form. A discrete function f (a discrete 1-form ω) is closed if $f \in \operatorname{Ker}(d_0)$ ($\omega \in \operatorname{Ker}(d_1)$).*

Also, we define the exactness of a discrete form. A discrete 1-form ω (a discrete 2-form σ) is exact if $\omega \in \operatorname{Im}(d_0)$ ($\sigma \in \operatorname{Im}(d_1)$).

Lemma 1.11. *For any discrete function $f \in \Omega^0$, we have*

$$d_1 d_0 f = 0.$$

Hence, exact 1-forms are closed, i.e. $\operatorname{Im} d_0 \subset \operatorname{Ker} d_1$.

The dimension of simplical homology of a closed surface is known to be twice of its genus from any standard algebraic topology text book (Bredon, 1993).

Theorem 1.12. *Suppose M is a closed oriented triangulated surface of genus g. Then,*

$$\dim(\operatorname{Ker} d_1 / \operatorname{Im} d_0) = 2g.$$

To prepare for Hodge decomposition theorem, dual discrete forms are introduced.

Definition 1.13. *A dual 0-form σ on a discrete surface M is a map $\sigma : F \to \mathbb{R}$. The space of all dual 0-forms is written as $\Omega^0(M^*)$.*

A dual 1-form ω on a discrete surface M is a map $\omega : \tilde{E} \to \mathbb{R}$ such that for any edge $e \in \tilde{E}$, $\omega(-e) = -\omega(e)$. In particular, for dual 1-form ω, we write $\omega(*e) := \omega(e)$ from now on to distinguish it from 1-form. The space of all dual 1-forms is written as $\Omega^1(M^*)$.

A dual 2-form f on a discrete surface M is a map $f : V \to \mathbb{R}$. The space of all dual 2-forms is written as $\Omega^2(M^*)$.

The reason for being called dual forms is that we would like to pair them with discrete differential forms to define a functional.

Lemma 1.14. *Notice that $\Omega^i(M^*)$ and $\Omega^{2-i}(M)$ share a common domain for $i = 1, 2, 3$. We can define their pairing respectively*

$$\begin{aligned} (\ ,\ ) : &\Omega^0(M^*) \times \Omega^2(M) \to \mathbb{R} \\ &\Omega^1(M^*) \times \Omega^1(M) \to \mathbb{R} \\ &\Omega^2(M^*) \times \Omega^0(M) \to \mathbb{R} \end{aligned}$$

by summation of their pointwise multiplicative products over F,E and V respectively. Such products are bilinear and nondegenerate. Hence, we have

$$\Omega^i(M^*) \cong \Omega^{2-i}(M)^*.$$

Similarly, we define the derivative of dual 0-forms and dual 1-forms.

Definition 1.15. *The derivative of dual 0-forms and dual 1-forms are defined by*

$$\begin{aligned}\partial_0 : \Omega^0(M^*) &\to \Omega^1(M^*)\\ f &\mapsto \partial f(e) = f(left(e)) - f(right(e))\end{aligned}$$

where left(e) and right(e) are the left and right faces of the edge e (Fig. 1.1). Similary, we define

$$\begin{aligned}\partial_1 : \Omega^1(M^*) &\to \Omega^2(M^*)\\ \omega &\mapsto \partial\omega(v_j) = \sum_{e_{ij}\in\tilde{E}:j} \omega(e_{ij}).\end{aligned}$$

Lemma 1.16. *For any discrete function f, 1-form ω, dual 1-form ω' and dual 0-form σ', we have*

$$\begin{aligned}(\omega', df) &= (\partial\omega', f),\\ (\sigma', d\omega) &= (\partial\sigma', \omega).\end{aligned}$$

Proof. By definitions, we have

$$\begin{aligned}(\omega', df) &= \sum_{e_{ij}\in E} \omega'(e_{ij})(f_j - f_i)\\ &= \sum_{j\in V}\Big(f_j \sum_{e_{ij}\in E:j} \omega'(e_{ij})\Big)\\ &= (\partial\omega', f)\end{aligned}$$

and

$$\begin{aligned}(\sigma', d\omega) &= \sum_{\phi_{ijk}\in F} \sigma'(\phi_{ijk})\big(\omega(e_{ij}) + \omega(e_{jk}) + \omega(e_{ki})\big)\\ &= \sum_{e_{ij}\in E} \omega(e_{ij})\big(\sigma'(left(e_{ij})) - \sigma'(right(e_{ij}))\big)\\ &= (\partial\sigma', \omega).\end{aligned}$$

□

Now we want to have identification between discrete forms and discrete dual forms. Let $T_i : \Omega^i(M) \to \Omega^{2-i}(M^*)$ be a symmetric invertible positive definite linear map. Then, we can define inner products

$$\begin{aligned}\langle\ ,\ \rangle : \Omega^0(M)\times\Omega^0(M) &\to \mathbb{R}\\ f_1, f_2 &\mapsto (T_0 f_1, f_2),\\ \Omega^1(M)\times\Omega^1(M) &\to \mathbb{R}\\ \omega_1, \omega_2 &\mapsto (T_1\omega_1, \omega_2),\\ \Omega^2(M)\times\Omega^0(M) &\to \mathbb{R}\\ \sigma_1, \sigma_2 &\mapsto (T_2\sigma_1, \sigma_2).\end{aligned}$$

The inner products defined are symmetric and nondegenerate.

Example 1. *Suppose a triangulated surface is equipped with a length function defined on edges. Then areas $A(\phi_{ijk})$ of each triangle can then be defined. We can take T_0,T_1 and T_2 be diagonal matrices with diagonal entries of the forms $\sum_{\phi_{ijk}\in F:j} A(\phi_{ijk})$, $(\cot\beta_k + \cot\beta_{\tilde{k}})/2$ where $\phi_{ijk}, \phi_{i\tilde{k}j} \in F$ and $1/A(\phi_{ijk})$ respectively. It is an example for the identifications between i-forms and dual i-forms.*

In the following, we assume the identifications T_i are fixed.

Lemma 1.17. *Under the inner products on $\Omega^i(M)$, we have*

$$\delta_2 := d_1^* = T_1^{-1}\partial_0 T_2,$$
$$\delta_1 := d_0^* = T_0^{-1}\partial_1 T_1.$$

Notice that $\delta_1 \circ \delta_2 = 0$.

Definition 1.18. *A harmonic 1-form ω is a 1-form which is closed (i.e. $d\omega = 0$) and co-closed (i.e. $\delta\omega = 0$). The space of harmonic 1-forms is denoted as $\mathcal{H}_1$.*

Theorem 1.19. *(Hodge Decomposition Theorem) Any discrete 1-form can be written uniquely as the sum of an exact 1-form, a co-exact 1-form and a harmonic 1-form. That means the space of all discrete 1-forms Ω^1 can be decomposed as*

$$\Omega^1 = d\Omega^0 \oplus \delta\Omega^2 \oplus \mathcal{H}.$$

PROOF. Notice that $\langle d\omega, \sigma\rangle = \langle \omega, \delta\sigma\rangle$, we then have

$$\operatorname{Ker} d_1 \subset (\delta\Omega^2)^\perp.$$

Given any $\omega \in (\delta\Omega^2)^\perp$, we know $\omega \in \Omega^1$ and $d\omega \in \Omega^2$. So, we get

$$0 = \langle \omega, \delta d\omega\rangle = \langle d\omega, d\omega\rangle.$$

It implies $d\omega = 0$ and $\omega \in \operatorname{Ker} d_1$. Hence $\operatorname{Ker} d_1 = (\delta\Omega^2)^\perp$. Similarly, we also have $\operatorname{Ker}\delta_1 = (d\Omega^0)^\perp$.

Since $d_0\Omega^0 \subset \operatorname{Ker} d_1$ and Ω^1 is finite dimensional vector space, we have

$$\begin{aligned}\Omega^1 =& \delta\Omega^2 \oplus \operatorname{Ker} d_1\\ =& \delta\Omega^2 \oplus \big(\operatorname{Ker} d_1 \cap \Omega^1\big)\\ =& \delta\Omega^2 \oplus \big(\operatorname{Ker} d_1 \cap (d_0\Omega^0 \oplus \operatorname{Ker}\delta_1)\big)\\ =& d\Omega^0 \oplus \delta\Omega^2 \oplus (\operatorname{Ker} d_1 \cap \operatorname{Ker}\delta_1)\\ =& d\Omega^0 \oplus \delta\Omega^2 \oplus \mathcal{H}.\end{aligned}$$

□

By the above decomposition and Theorem 1.12, we know the dimension of harmonic 1-forms.

Corollary 1.20.

$$\dim\mathcal{H}^1 = \dim(\operatorname{Ker} d_1/\operatorname{Im} d_0) = 2g$$

3. A One-dimensional Analogue

Deformations of plane curve serve as a one-dimensional analogue of infinitesimal conformal transformations. For surfaces in Euclidean space, their metric and Gaussian curvature are intrinsic information. They are not enough to determine immersions. Extrinsic information is needed, such as mean curvature. For plane curves, curvature is extrinsic. This section studies how curvatures determine the geometry of smooth and discrete curves. It would be analogous to smooth and triangulated surfaces.

In subsection 3.1, deformations of smooth plane curves are considered. For a given length L, a smooth closed plane curve γ is uniquely determined by its curvature κ up to an euclidean transformation. Such curvature functions parametrize the shape space of smooth closed plane curves. We would like to see how the curvature functions change under the deformation of γ in order to identify the tangent spaces of the shape space.

Analogous results in polygons are studied in Subsection 3.2.

3.1. Smooth Plane Curves. We will identify $\mathbb{R}^2$ as $\mathbb{C}$ in the following. Let $\gamma : \mathbb{R} \to \mathbb{C}$ be a smooth arc-length parametrized closed plane curve with length L. So, we have $\gamma(s+L) = \gamma(s)$ and $\gamma' = T$ with $|T| = 1$. Define the unit normal N of γ as $N := -JT$, where J denote the anti-clockwise $\frac{\pi}{2}$-rotation. The curvature κ is defined by $N' = \kappa T$.

The above definition implies there exist a function $\alpha : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} T(s) =& e^{i\alpha(s)}, \\ N(s) =& -ie^{i\alpha(s)}, \\ \kappa(s) =& \alpha'(s). \end{aligned}$$

Denote H as the euclidean vector space of L-periodic C^∞ function, $\mathcal{M} \subset H$ as the subset of H that can be realized as the curvature of some closed curve γ. By translation and rotation, we can assume $\gamma(0)$ is the origin and $T(0) = 1$. So given $\kappa \in M$, we have

$$\alpha(s) := \int_0^s \kappa, \tag{1.1}$$

$$T(s) = e^{i\alpha(s)}, \tag{1.2}$$

$$\gamma(s) = \int_0^s T. \tag{1.3}$$

Since γ is L-periodic, we have

$$\begin{aligned} T(s) =& T(s+L), \\ \gamma(s) =& \gamma(s+L) \end{aligned}$$

and hence

$$\begin{aligned} \int_0^L \kappa =& \alpha(L) - \alpha(0) \in 2\pi\mathbb{Z}, \\ \int_0^L T =& \gamma(L) - \gamma(0) = 0. \end{aligned}$$

Notice that given κ satisfying the above two equations, we can construct a closed plane curve by (1.1),(1.2),(1.3) with a prescribed curvature κ.

Lemma 1.21. *The space of L-periodic C^∞ functions which can be realized as the curvature of some arc-length parametrized closed curve is*

$$\mathcal{M} = \left\{ \kappa \in H \;\middle|\; \int_0^L \kappa \in 2\pi\mathbb{Z}, \int_0^L \cos(\int_0^s \kappa)ds = 0, \int_0^L \sin(\int_0^s \kappa)ds = 0 \right\}.$$

Hence, we know $\mathcal{M} = \bigcup_{n\in\mathbb{Z}} \mathcal{M}_n$ where

$$\mathcal{M}_n := \left\{ \kappa \in H \;\middle|\; \int_0^L \kappa = 2\pi n, \int_0^L \cos(\int_0^s \kappa)ds = 0, \int_0^L \sin(\int_0^s \kappa)ds = 0 \right\}$$

and $\mathcal{M}_n$'s are disjoint.

Theorem 1.22. *Let a positive real number L be fixed. Denote $\mathcal{M}$ as the space of the curvature functions of closed plane curves with length L. Then $\mathcal{M}$ is a manifold and if γ is a curve with curvature $\kappa \in \mathcal{M}$, the tangent space at κ is given by*

$$T_\kappa \mathcal{M} = \left\{ \dot{\kappa} \in H \;\Big|\; \int_0^L \dot{\kappa} = 0, \int_o^L \dot{\kappa}\gamma = 0 \right\}$$

where H is denoted as the space of L-periodic C^∞ functions.

Proof. Let $\kappa \in \mathcal{M}_n$. Define a function

$$\begin{aligned} F_n &: H \to \mathbb{R}^3 \\ &\kappa \mapsto \left(\int_0^L \kappa - 2\pi n, \operatorname{Re}(\gamma(L)), \operatorname{Im}(\gamma(L))\right) \end{aligned}$$

where γ is given by (1.3). Consider a variation $\kappa_t \in H$ of $\kappa = \kappa_0$. Then, using equations (1.1),(1.2),(1.3), we have

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \gamma_t(L) &= \int_0^L \dot{T}(s)ds \\ &= \int_0^L \dot{\alpha}(s)(-\sin\alpha(s), \cos\alpha(s)) \\ &= -J \int_0^L \dot{\alpha}(s)T(s)) \\ &= -J \int_0^L \dot{\alpha}(s)(\frac{d}{ds}\gamma(s))ds \\ &= J \int_0^L (\frac{d}{ds}\dot{\alpha}(s))\gamma(s)ds \\ &= J \int_0^L \dot{\kappa}(s)\gamma(s)ds \\ &= \int_0^L \dot{\kappa}(s)(i\gamma_1(s) - \gamma_2(s))ds. \end{aligned}$$

where $\gamma_1 = \operatorname{Re}(\gamma)$ and $\gamma_2 = \operatorname{Im}\gamma$. On the other hand,

$$\frac{d}{dt}\Big|_{t=0} \int_0^L \kappa - 2\pi n = \int_0^L \dot{\kappa}.$$

Hence,

$$d_\kappa F_n(\dot{\kappa}) = \left(\int_0^L \dot{\kappa}, -\int_0^L \dot{\kappa}(s)\gamma_2(s)ds, \int_0^L \dot{\kappa}(s)\gamma_1(s)ds\right).$$

Assume $d_\kappa F$ is not surjective. Then there exists constants $a_1, a_2, b \in \mathbb{R}$ such that

$$0 = a_1 \int_0^L \dot{\kappa}\gamma_1 + a_2 \int_0^L \dot{\kappa}\gamma_2 - b \int_0^L \dot{\kappa} = \int_0^L \dot{\kappa}(a_1\gamma_1 + a_2\gamma_2 - b)$$

for any L-periodic function $\dot{\kappa}$. It leads to a contradiction that $(a_1\gamma_1 + a_2\gamma_2 - b) \equiv 0$ and the closed curve γ is contained in a straight line.

Thus, $d_\kappa F_n$ is surjective and by implicit function theorem we conclude that $\mathcal{M} = F^{-1}(0)$ is a manifold. And the tangent space at κ is $\operatorname{Ker}(d_\kappa F_n)$. □

3.2. Discrete Plane Curves. We investigate the space of closed polygons with fixed edge lengths. We start with some notations.

Let $(\gamma_0, \ldots, \gamma_{n-1})$ be a polygon in the complex plane $\mathbb{C}$ and consider all indices module n. Then there are unique real numbers $\ell_0, \ldots, \ell_{n-1} > 0$ and unit vectors $T_0, \ldots, T_{n-1} \in S^1$ such that the edge vectors of γ have the form

$$\gamma_{j+1} - \gamma_j = \ell_j T_j.$$

Definition 1.23. *A polygon $(\gamma_0, \ldots, \gamma_n)$ is regular if*

$$T_{j+1} \neq -T_j \quad \forall i.$$

For a regular polygon, there exists unique real numbers $\kappa_1, \ldots, \kappa_{n-1}$ such that

$$\begin{aligned} -\pi < \kappa_j < \pi, \\ T_j = e^{i\kappa_j} T_{j-1}. \end{aligned}$$

Hence,

$$\begin{aligned} \alpha_j :=& \sum_{k=1}^{j} \kappa_j, \\ T_j =& e^{i\alpha_j} T_0, \\ \gamma_j =& \gamma_0 + \sum_{k=0}^{j-1} \ell_j T_j. \end{aligned} \tag{1.4}$$

Since the index of γ is n-periodic, we have

$$\begin{aligned} T_s =& T_{s+n}, \\ \gamma_s =& \gamma_{s+n} \end{aligned}$$

and so

$$\begin{aligned} \sum_{k=1}^{n} \kappa = \alpha_n = 2\pi r \quad \text{for some } r \in \mathbb{Z}, \\ \sum_{k=0}^{n-1} \ell_j T_j = \gamma_n - \gamma_0 = 0. \end{aligned}$$

For a given collection of edge lengths $\ell_0, \ldots, \ell_{n-1}$ and κ satisfying the above two equations, there exists a discrete curve γ with κ by the construction of (1.4) with arbitrary initial direction T_0 and initial position γ_0. Hence, it is unique up to rotation and translation.

In the following we assume to have a fixed sequence of edge lengths $\ell_0, \ldots, \ell_{n-1}$ which can be realized by some regular closed polygon. Denote $M \subset (-\pi, \pi)^n$ as the set containing all elements which can be realized as the curvature of some regular closed polygon γ. The above argument implies the closed polygon is unique up to translation and rotation. In particular, the polygon can be assumed to have $\gamma_0 = 0$ and $T_0 = 1$. We have

Lemma 1.24.

$$\mathcal{M} = \left\{ \kappa \in (-\pi, \pi)^n \;\Big|\; \sum_{k=1}^{n} \kappa \in 2\pi\mathbb{Z}, \sum_{k=0}^{n-1} \ell_j T_j = 0 \right\}.$$

So, $\mathcal{M} = \bigcup_{r \in \mathbb{Z}} \mathcal{M}_r$ where

$$\mathcal{M}_r := \left\{ \kappa \in (-\pi, \pi)^n \;\Big|\; \sum_{k=1}^{n} \kappa = 2r\pi, \sum_{k=0}^{n-1} \ell_j T_j = 0 \right\}$$

and $\mathcal{M}_n$'s are disjoint.

Theorem 1.25. *Suppose a length sequence $\ell_0, \dots, \ell_{n-1}$ can be realized by some regular closed polygon. Denote $\mathcal{M}$ as the space of the curvature functions of closed planar polygons with the fixed length sequence $\ell_0, \dots, \ell_{n-1}$. Then $\mathcal{M}$ is a manifold and if γ is a regular closed polygon corresponding to $\kappa \in \mathcal{M}$, the tangent space at κ is given by*

$$T_\kappa \mathcal{M} = \left\{ \kappa \in (-\pi, \pi)^n \;\middle|\; \sum_{m=0}^{n-1} \dot{\kappa} = 0, \sum_{m=0}^{n-1} \dot{\kappa}_m \gamma_m = 0 \right\}.$$

Proof. Let $\kappa \in \mathcal{M}_r$. And γ be the curve with curvature κ, $\gamma_0 = 0$ and $T_0 = 1$. Define a function

$$\begin{aligned} F_r : (-\pi, \pi)^n &\to \mathbb{R}^3 \\ \kappa &\mapsto (\sum_{m=0}^{n-1} \kappa - 2\pi r, \operatorname{Re}(\gamma_n), \operatorname{Im}(\gamma_n)). \end{aligned}$$

Consider a variation $\kappa(t) \in (-\pi, \pi)^n$ of $\kappa(0) = \kappa$. Then,

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \gamma_n(t) &= \sum_{j=0}^{n-1} \ell_j \dot{T}_j \\ &= \sum_{j=0}^{n-1} \ell_j \, i \sum_{m=1}^{j} \dot{\kappa}_m \, e^{i \sum_{k=1}^{j} \kappa_k} T_0 \\ &= i \sum_{m=1}^{n-1} \dot{\kappa}_m \sum_{j=m}^{n-1} \ell_j e^{i \sum_{k=1}^{j} \kappa_k} T_0 \\ &= i \sum_{m=1}^{n-1} \dot{\kappa}_m (\gamma_n - \gamma_m) \\ &= -i \sum_{m=0}^{n-1} \dot{\kappa}_m \gamma_m + i \sum_{m=0}^{n-1} \dot{\kappa}_m \gamma_0 \\ &= -i \sum_{m=0}^{n-1} \dot{\kappa}_m \gamma_m. \end{aligned}$$

On the other hand,

$$\frac{d}{dt}\Big|_{t=0} \sum_{m=0}^{n-1} \kappa - 2\pi n = \sum_{m=0}^{n-1} \dot{\kappa}.$$

Hence,

$$d_\kappa F_r(\dot{\kappa}) = (\sum_{m=0}^{n-1} \dot{\kappa}, \sum_{m=0}^{n-1} \dot{\kappa}_m \operatorname{Im}(\gamma_m), -\sum_{m=0}^{n-1} \dot{\kappa}_m \operatorname{Re}(\gamma_m)).$$

Assume $d_\kappa F_r$ is not surjective. Then there exists constants $a_1, a_2, b \in \mathbb{R}$ such that

$$\begin{aligned} 0 &= a_1 \sum_{m=0}^{n-1} \dot{\kappa}_m \operatorname{Re}(\gamma_m) + a_2 \sum_{m=0}^{n-1} \dot{\kappa}_m \operatorname{Im}(\gamma_m) - b \sum_{m=0}^{n-1} \dot{\kappa} \\ &= \sum_{m=0}^{n-1} \dot{\kappa}_m (a_1 \operatorname{Re}(\gamma_m) + a_2 \operatorname{Im}(\gamma_m) - b) \end{aligned}$$

for all $\dot{\kappa} \in (-\pi, \pi)^n$. It leads to the contradiction that the closed regular polygon γ is contained in a straight line.

Thus, $d_\kappa F_r$ is surjective and by implicit function theorem we conclude that $\mathcal{M}_r = F_r^{-1}(0)$ is a manifold. And the tangent space at κ is $\operatorname{Ker}(d_\kappa F_r)$. $\square$

CHAPTER 2

Conformal Deformations of Smooth Surfaces

The quaternionic approach provides a coordinate free way to study conformal immersions regularly homotopic to a given one. Generically, such a space is parameterized by the mean curvature half-density $H|df|$ which is related to the eigenvalues of the quaternionic Dirac Operator D. It is used to study Bonnet problems and Willmore surfaces (Pedit and Pinkall, 1998; Richter, 1997).

The quaternionic theory applied to differential geometry (Kamberov et al., 1998) is reviewed in Section 1-3 in order to compare with the discrete analogues. We derive the quaternionic Dirac operator from an integrability condition and study its connection to mean curvature half-densities.

Then we focus on infinitesimal conformal deformations and their complement in the space of all half-densities. Sections 4-6 are a collection of results from the blog "Discrete Spin Geometry" where Theorem 2.21 in Section 5 is generalized to surfaces of genus $g > 1$. In particular, Theorem 2.22 is essential to the derivation of the discrete Dirac operator in Chapter 3.

1. Spin Transformations

We start with the definition of a complex structure.

Definition 2.1. *A Riemann surface is a real two-dimensional manifold M equipped with an automorphism $J \in \mathrm{Aut}(TM)$ such that $J^2 = -1$. We call J a complex structure of M and the tangent spaces of M has the structure of a complex vector space naturally,*

$$\forall z = x + iy \in \mathbb{C} \text{ and } X \in T_pM, \quad zX := (x + yJ_p)X \in T_pM.$$

Definition 2.2. *The Hodge star operator $*$ is an operator on K-valued 1-form ζ, where $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ such that*

$$*\zeta = \zeta \circ J.$$

With a complex structure, 2-forms can be identified as complex quadratic forms. It is noticed that for any $\mathbb{H}$-valued 2-form $\omega : TM \to \mathbb{H}$,

$$\begin{aligned}\omega\big((a + bJ)X, J(a + bJ)X\big) =& (a^2 + b^2)\omega(X, JX)\\ =& |a + bJ|^2\omega(X, JX).\end{aligned}$$

So, a 2-forms is identified as a complex quadratic form via

$$\forall X \in TM, \quad q(X) = \omega(X, JX),$$

which satisfies

$$\forall X \in TM \text{ and } z \in \mathbb{C}, \quad q(zX) = |z|^2 q(X).$$

Such correspondence is bijective, since for any tangent vectors X and Y,

$$\omega(X, Y) = \frac{1}{2}\big(q(Y) + q(X) - q(X + JY)\big).$$

In particular, the space of real-valued complex quadratic forms on M is denoted as densities D, which under the above identification are real-valued 2-forms $\Omega^2(M,\mathbb{R})$. We then introduce half-densities

$$D^{\frac{1}{2}} = \{\delta : TM \to \mathbb{R} | \delta(zX) = |z|\delta(X) \quad \forall z \in \mathbb{C}\}.$$

We now focus on immersions of a surface into Euclidean space. In the following, we identify $\mathbb{R}^3$ as $\operatorname{Im}\mathbb{H}$. Given an immersion f of a surface M into Euclidean space, there is an induced metric on M. For any tangent vectors $X, Y \in T_pM$,

$$f^* g_{\mathbb{R}^3}(X,Y) = \langle df(X), df(Y)\rangle.$$

Definition 2.3. *An immersion f of a Riemann surface is conformal if it satisfies for any tangent vector X,*

$$\begin{aligned}\langle df(X), df(X)\rangle &= \langle *df(X), *df(X)\rangle,\\ \langle df(X), *df(X)\rangle &= 0.\end{aligned}$$

Remark 2.4. *If f is a conformal immersion, then $|df|^2$ is a density and $|df|$ is a half-density.*

Definition 2.5. *Two Riemannian metrics g and $\tilde{g}$ of M are conformally equivalent if there exists a real-valued function $u : M \to \mathbb{R}$ such that*

$$\tilde{g} = e^u g.$$

Lemma 2.6. *The metrics induced by two conformal immersions f and $\tilde{f}$ are conformally equivalent.*

Proof. Take $u = \ln(\frac{|d\tilde{f}(X)|}{|df(X)|})$. Then,

$$\tilde{f}^* g_{\mathbb{R}^3} = e^u f^* g_{\mathbb{R}^3}.$$

□

Definition 2.7. *Given two immersions f and $\tilde{f}$ of a surface M into $\mathbb{R}^3 \cong \operatorname{Im}\mathbb{H}$, $\tilde{f}$ is a spin transformation of f if there exists a quaternion-valued function $\lambda : M \to \mathbb{H}\backslash\{0\}$ such that*

$$d\tilde{f} = \bar{\lambda} df \lambda. \tag{2.1}$$

Lemma 2.8. *If f is a conformal transformation of a Riemann surface M and $\tilde{f}$ is its spin transformation, then $\tilde{f}$ is also a conformal immersion of M. And the metrics induced by f and $\tilde{f}$ are conformally equivalent.*

Proof. Notice that

$$\begin{aligned}\langle d\tilde{f}, *d\tilde{f}\rangle &= \langle \bar{\lambda} df \lambda, \bar{\lambda} * df \lambda\rangle\\ &= Re(\bar{\lambda} df \lambda \overline{\bar{\lambda} * df \lambda})\\ &= |\lambda|^4 \langle df, *df\rangle\\ &= 0.\end{aligned}$$

Similar calculation gives

$$\langle d\tilde{f}, d\tilde{f}\rangle = \langle *d\tilde{f}, *d\tilde{f}\rangle = |\lambda|^4 \langle df, df\rangle.$$

□

We then study a necessary condition on λ for the existence of a spin transformation. Suppose $\tilde{f}$ is a spin transformation of f, then by definition there exists a $\mathbb{H}$-valued function $\lambda : M \to \mathbb{H}\backslash\{0\}$ such that

$$\begin{aligned}d\tilde{f} &= \bar{\lambda} df \lambda\\ 0 &= d(d\tilde{f})\end{aligned}$$

$$\begin{aligned} &= d(\bar{\lambda} df \lambda) \\ &= d\bar{\lambda} df \lambda - \bar{\lambda} df d\lambda \\ &= \overline{\bar{\lambda} df d\lambda} - \bar{\lambda} df d\lambda \\ &= -2\,\mathrm{Im}(\bar{\lambda} df d\lambda). \end{aligned}$$

Hence, there exists a real valued function $\rho : M \to \mathbb{R}$ such that

$$\begin{aligned} \bar{\lambda} df d\lambda &= -\rho|\lambda|^2|df|^2 \\ df d\lambda &= -\rho\lambda|df|^2 \\ \mathrm{D}\,\lambda := -\frac{df d\lambda}{|df|^2} &= \rho\lambda. \end{aligned}$$

Definition 2.9. *Denote the space of quaternionic functions on M as $\Gamma(M, \mathbb{H})$. The differential operator $\mathrm{D} : \Gamma(M, \mathbb{H}) \to \Gamma(M, \mathbb{H})$ defined above is called quaternionic Dirac operator.*

We conclude the above calculation by the following lemma.

Theorem 2.10. *Given a conformal immersion $f : M \to \mathrm{Im}(\mathbb{H})$, if a quaternion-valued function $\lambda : M \to \mathbb{H}\backslash\{0\}$ induces a spin transformation $\tilde{f}$ of f given by*

$$d\tilde{f} = \bar{\lambda} df \lambda,$$

then there exists $\rho : M \to \mathbb{R}$ such that

$$\mathrm{D}\,\lambda = \rho\lambda. \tag{2.2}$$

On the other hand, given $\lambda : M \to \mathbb{H}$ satisfying (2.2), then $\bar{\lambda} df \lambda$ is a closed 1-form.

The above lemma imposes a necessary condition on λ for the existence of a spin transformation. In general, it is not sufficient since a closed 1-form may not be exact. But for simply connected surfaces, a closed 1-form is also exact.

Corollary 2.11. *If M is simply connected, the converse of Theorem 2.10 is true, i.e. a quaternion-valued function $\lambda : M \to \mathbb{H}\backslash\{0\}$ induces a spin transformation of f if and only if there exists $\rho : M \to \mathbb{R}$ such that*

$$\mathrm{D}\,\lambda = \rho\lambda.$$

2. Mean Curvature Half-Density

In this section, we look at how geometric quantities change under a spin transformation. And the geometric meaning of $\rho|df|$ as a change of mean curvature half-density is shown.

Lemma 2.12. *Suppose $f : M \to \mathrm{Im}\,\mathbb{H}$ is a conformal immersion of a Riemann surface M. We denote its normal vector field as N and its mean curvature as H. Then, we have*

$$df \wedge dN = 2HN|df|^2.$$

Proof. Denote A as the shape operator of f, which is defined via

$$dN = df \circ \mathrm{A}\,.$$

Let X be any unit vector. Then

$$\begin{aligned} df dN(X, JX) =& df(X)dN(JX) - df(JX)dN(X) \\ =& df(X)df(\mathrm{A}\,JX) - df(JX)df(\mathrm{A}\,X) \\ =& -\langle X, \mathrm{A}\,JX\rangle + \langle JX, AX\rangle + (\langle JX, \mathrm{A}\,JX\rangle + \langle X, AX\rangle)N \\ =& 2HN. \end{aligned}$$

Hence,

$$df\,dN = 2HN|df|^2.$$

□

Theorem 2.13. *Suppose* $f : M \to \operatorname{Im}\mathbb{H}$ *is a conformal immersion and* $\tilde{f}$ *is its spin transformation given by* $\lambda : M \to \mathbb{H}\backslash\{0\}$*, which satisfies*

$$\mathrm{D}\,\lambda = \rho\lambda.$$

Then,

$$\begin{aligned}|d\tilde{f}|^2 &= |\lambda|^4|df|^2,\\ \tilde{N} &= \lambda^{-1}N\lambda,\\ \tilde{H}|d\tilde{f}| &= H|df| + \rho|df|.\end{aligned}$$

Proof. Followed from Lemma 2.8, $\tilde{f}$ is a conformal immersion and

$$|d\tilde{f}|^2 = |\lambda|^4|df|^2.$$

Let X be any nonzero vector. Then

$$\begin{aligned}\tilde{N} =& \frac{d\tilde{f}(X)d\tilde{f}(JX)}{|d\tilde{f}(X)||d\tilde{f}(JX)|}\\ =& \frac{\lambda^{-1}df(X)df(JX)\lambda}{|df(X)||df(JX)|}\\ =& \lambda^{-1}N\lambda.\end{aligned}$$

Also,

$$\begin{aligned}d\tilde{f}d\tilde{N} =& \overline{\lambda}df\lambda d(\lambda^{-1}N\lambda)\\ =& \overline{\lambda}df\lambda(d\lambda^{-1})N\lambda + \overline{\lambda}df\,dN\lambda + \overline{\lambda}df\,Nd\lambda\\ =& -\overline{\lambda}df\,d\lambda\lambda^{-1}N\lambda + 2H\overline{\lambda}N\lambda|df|^2 - \overline{\lambda}Ndf\,d\lambda\\ =& 2(\rho+H)|df|^2\overline{\lambda}N\lambda\\ =& 2(\rho+H)|df|^2|\lambda|^2\tilde{N}.\end{aligned}$$

On the other hand,

$$d\tilde{f}d\tilde{N} = 2\tilde{H}\tilde{N}|d\tilde{f}|^2.$$

Hence, equating the above two equations, we have

$$\begin{aligned}\tilde{H}|d\tilde{f}|^2 =& (H+\rho)|\lambda|^2|df|^2\\ \tilde{H}|d\tilde{f}| =& H|df| + \rho|df|.\end{aligned}$$

□

Hence, we call $H|df|$ the mean curvature half-density of f and $\rho|df|$ the change of mean curvature half-density corresponding to the spin transformation given by λ.

3. The Quaternionic Dirac Operator

Properties of the quaternionic Dirac operator D are reviewed here in order to compare with the discrete Dirac operator in Chapter 3.

Lemma 2.14. *Suppose* M *is a closed Riemann surface. Then the quaternionic Dirac operator* $\mathrm{D} : \Gamma(M,\mathbb{H}) \to \Gamma(M,\mathbb{H})$ *is elliptic and self adjoint with respect to* L^2 *product. Hence,*

$$C^\infty(M,\mathbb{H}) = \operatorname{Ker}(\mathrm{D}) \oplus \operatorname{Im}(\mathrm{D}).$$

Proof. We first show that it is self adjoint. Let $\lambda_1, \lambda_2 \in C^\infty(M, \mathbb{H})$.

$$\begin{aligned}\int_M \langle \mathrm{D}\,\lambda_1, \lambda_2\rangle |df|^2 =& Re\Big(\int_M (-df d\lambda_1)\overline{\lambda}_2\Big)\\ =& Re\Big(\int_M df \lambda_1 \overline{d\lambda_2}\Big)\\ =& -Re\Big(\int_M \lambda_1 \overline{d\lambda_2} df\Big)\\ =& -Re\Big(\int_M \lambda_1 \overline{df d\lambda_2}\Big)\\ =& \int_M \langle \lambda_1, \mathrm{D}\,\lambda_2\rangle |df|^2.\end{aligned}$$

We then show that it is elliptic. The definition of ellipticity and hence the decomposition is referred to Lax (2002). Let $x \in M, \zeta \in T_x^*M$ and $h \in C^\infty(M, \mathbb{R})$ such that $h(x) = 0$ and $dh = \zeta$. Suppose $\lambda \in C^\infty(M, \mathbb{H})$. Then the principal symbol of D at ζ is given by

$$\begin{aligned}\sigma_1(\mathrm{D}, \zeta) \cdot \lambda(0) :=& \mathrm{D}(h\lambda)|_x\\ =& -\frac{df \wedge \zeta}{|df|^2}\lambda(0).\end{aligned}$$

For $\zeta \neq 0$, we have $\sigma_1(\mathrm{D}, \zeta) = -\frac{df\wedge\zeta}{|df|^2} \neq 0$ since $f : M \to \mathrm{Im}(\mathbb{H})$ is an immersion. Hence it is elliptic.

□

An observation from the definition of the quaternionic Dirac operator leads to the following.

Corollary 2.15. *Constant quaternion-valued functions are in the kernel of* D. *Hence*

$$\dim(\mathrm{Ker}\,\mathrm{D}) \geq 4.$$

4. Infinitesimal Conformal Deformations

Starting from this section, we look at infinitesimal conformal deformations, which is the focus of this thesis.

Suppose a 1-parameter family of spin transformations of $f : M \to \mathbb{R}^3$ is given by λ_t with $\lambda(0) \equiv 1$. Then there exists a 1-parameter family of $\mathbb{R}$-valued functions $\rho_t : M \to \mathbb{R}$ with $\rho_{t=0} \equiv 0$ such that

$$\mathrm{D}\,\lambda_t = \rho_t \lambda_t.$$

From the derivation of the Dirac operator D, the above condition is equivalent to $\overline{\lambda_t} df \lambda_t$ being a closed 1-form. Differentiating both sides with respect to t and evaluating at $t = 0$, we get

$$\begin{aligned}\mathrm{D}\,\dot{\lambda} &= \dot{\rho}\lambda_{t=0} + \rho_{t=0}\dot{\lambda}\\ &= \dot{\rho}\end{aligned}$$

which implies that $\mathrm{Im}(2df\dot{\lambda})$ is a closed 1-form.

Theorem 2.16. *Under the infinitesimal conformal deformation given by* $\dot{\lambda} : M \to \mathbb{H}$ *satisfying* $\mathrm{D}\,\dot{\lambda} = \dot{\rho}$, *we have*

$$\int_M \dot{\rho}|df|^2 = 0.$$

PROOF.

$$\begin{aligned}\int_M \dot\rho|df|^2 &= \int_M \mathrm{D}\,\dot\lambda|df|^2\\ &= \int_M df\wedge d\dot\lambda\\ &= 0.\end{aligned}$$

□

The above lemma is a necessary condition for infinitesimal conformal deformations. It is not sufficient in general. But for surfaces with genus 0, it is sufficient, since a closed 1-form implies its exactness.

Corollary 2.17. *Suppose $f : S^2 \to \mathbb{R}^3$ is an immersion of the sphere S^2 with* $\dim(\mathrm{Ker}(\mathrm{D})) = 4$. *Given any $\dot\rho : M \to \mathbb{R}$, then there exists an infinitesimal conformal deformation given by $\dot\lambda$ satisfying $\mathrm{D}\,\dot\lambda = \dot\rho$ if and only if $\int_M \dot\rho|df|^2 = 0$*

PROOF. One direction of the statement is proved in the previous lemma. We proceed with another direction. The assumption $\dim(\mathrm{Ker}(\mathrm{D})) = 4$ implies Ker(D) consists of constants quaternion-valued functions. $\int_M \dot\rho|df|^2 = 0$ implies $\dot\rho$ is L^2 perpendicular to Ker(D). By the decomposition in Lemma 2.14, we know $\dot\rho \in \mathrm{Im}\,\mathrm{D}$. Hence, there exists $\dot\lambda : M \to \mathbb{H}$ such that $\mathrm{D}\,\dot\lambda = \dot\rho$. From the derivation of the Dirac operator, it implies $(\dot{\overline{\lambda df\lambda}})$ is a closed 1-form. For surfaces with genus 0, it implies exactness. □

5. Infinitesimal Conformal Deformations of High Genus Surfaces

For high genus surfaces, $\mathbb{H}$-valued functions $\dot\lambda$ satisfying $\mathrm{D}\,\dot\lambda = \dot\rho$ may not preserve the exactness of df. To ensure the exactness, there are other constraints. We first look at restrictions on $\dot\rho$.

The condition $\mathrm{D}\,\dot\lambda = \dot\rho$ implies the 1-form $\mathrm{Im}(df\,\dot\lambda)$ is closed. If $\beta_1, \dots, \beta_{2g}$ is a basis for harmonic 1-forms, then

$$\mathrm{Im}(df\,\dot\lambda)$$

is exact if and only if

$$0 = \int_M *\beta_i \wedge \mathrm{Im}\, df\,\dot\lambda = \mathrm{Im}\int_M *\beta_i \wedge df\,\dot\lambda$$

for $i = 1, \dots, 2g$. If $X, JX \in T_pM$ form an orthonormal basis, then

$$\begin{aligned}(*\beta_i \wedge df)(X, JX) &= \beta_i(JX)df(JX) + \beta_i(X)df(X)\\ &= df(\beta_i(X)X + \beta_i(JX)df(JX)) =: df(Y_i).\end{aligned}$$

The vector field Y_i is the harmonic vector field corresponding to the harmonic 1-form β_i. By denoting the $\mathbb{R}^3$-valued functions as $y_i = df(Y_i)$, we have

$$*\beta_i \wedge df = y_i|df|^2$$

and the exactness condition becomes

$$\mathrm{Im}\int_M y_i\dot\lambda|df|^2 = 0 \quad \forall i = 1, 2, \dots, 2g.$$

We have

Lemma 2.18.

$$\int_M y_i|df|^2 = 0.$$

PROOF. Notice that $y_i|df|^2 = *\beta_i \wedge df$ is a exact 2-form, its integral over the whole surface M hence is zero. □

If the kernel of D is of dimension 4, then there exist quaternion-valued functions $\mu_i : M \to \mathbb{H}$ satisfying

$$\mathrm{D}\,\mu_i = y_i.$$

The exactness conditions become equivalent to say that for all constant quaternions $a \in \mathrm{Im}\, H$ and for $i = 1, 2, \ldots, 2g$,

$$\begin{aligned}
0 &= \mathrm{Re}\left(\int_M y_i \dot{\lambda} a\right) \\
&= \mathrm{Re}\left(\int_M y_i a \overline{\dot{\lambda}}\right) \\
&= \int_M \langle y_i a, \dot{\lambda}\rangle \\
&= \int_M \langle \mathrm{D}\,\mu_i a, \dot{\lambda}\rangle \\
&= \int_M \langle \mu_i a, \mathrm{D}\,\dot{\lambda}\rangle \\
&= \int_M \langle \mu_i a, \dot{\rho}\rangle \\
&= -\int_M \langle \mu_i, a\rangle \dot{\rho}.
\end{aligned}$$

The following theorem describes the tangent space of conformal immersions.

Theorem 2.19. *Suppose M is a closed Riemann surface of genus g and $f : M \to \mathbb{R}^3$ is a conformal immersion with $\dim(\mathrm{Ker}\,\mathrm{D}) = 4$. Let a function $\dot{\lambda} : M \to \mathbb{H}$ satisfy $D\dot{\lambda} = \dot{\rho}$ for some real valued function $\dot{\rho}$. Then, there is an infinitesimal conformal deformation given by $\dot{\lambda}$, i.e. $\mathrm{Im}(df\,\dot{\lambda})$ is exact, if and only if $\dot{\rho}$ is L^2-orthogonal to all three imaginary components of each of the $2g$ functions μ_i and the constant function* **1**.

Hence, the vector space

$$\{\dot{\rho} \in C^\infty(M) | \int_M \dot{\rho} = \int_M \langle \mu_l, i\rangle \dot{\rho} = \int_M \langle \mu_l, j\rangle \dot{\rho} = \int_M \langle \mu_l, k\rangle \dot{\rho} = 0 \text{ for } l = 1, \ldots, 2g\}$$

is the space of all nontrivial infinitesimal conformal deformations of f, i.e. up to Euclidean transformations.

These $6g + 1$ equations are not necessary linearly independent over $\mathbb{R}$. For special surfaces, they are indeed linearly dependent.

Definition 2.20. *An immersion $f : M \to \mathbb{R}^3$ is isothermal if there exists a $\mathrm{Im}(\mathbb{H})$-valued closed 1-form τ such that*

$$df \wedge \tau = 0.$$

If the immersion f is isothermal and N is denoted as the gauss map of f, then locally there exists an immersion f^* which is $-N$-conformal. A more detailed discussion can be found in Kamberov et al. (1998); Richter (1997). If an immersion has $\dim(\mathrm{Ker}(\mathrm{D})) = 4$, being isothermal is equivalent to those $6g+1$ equations being linearly dependent.

Theorem 2.21. *Suppose M is a closed Riemann surface of genus g and $f : M \to \mathbb{R}^3$ is a conformal immersion with $\dim(\mathrm{Ker}\,\mathrm{D}) = 4$. The tangent space of all conformal immersions at f is of co-dimension less than or equal to $6g+1$. The inequality is strict if and only if the immersion f is isothermal.*

Proof. Let $a_1, b_1, a_2, \ldots, b_g$ be a canonical basis of homology and $\omega_1, \ldots, \omega_{2g}$ be a basis of harmonic 1-forms such that $\int_{a_j} \omega_i = \int_{b_j} \omega_{g+i} = \delta_{ij}$ and $\int_{a_j} \omega_{g+i} =$

$\int_{b_j} \omega_i = 0$. The condition $\dim(\operatorname{Ker} \mathrm{D}) = 4$ implies there exists $\mu_i : M \to \mathbb{H}$ such that $-df \wedge d\mu_i = df \wedge \omega_i$.

Suppose the 6g+1 functions, which are the three imaginary components of each μ_i and the constant function **1**, are linearly dependent. Then there exists constants $v_1, v_2, \ldots, v_{2g} \in \operatorname{Im}(\mathbb{H})$ and $c \in \mathbb{R}$ such that they are not all zero and

$$\langle \mu_1, v_1 \rangle + \langle \mu_2, v_2 \rangle + \cdots + \langle \mu_{2g}, v_{2g} \rangle = c.$$

With out loss of generality, we assume $c = 0$ since we can replace some μ_i by $\tilde{\mu}_i := \mu_i - \frac{cv_i}{\langle v_i, v_i \rangle}$ if v_i is not zero. Hence,

$$\operatorname{Re} \sum_{i=1}^{2g} \mu_i v_i \equiv 0.$$

Taking exterior derivative,

$$\operatorname{Re} \sum_{i=1}^{2g} d\mu_i v_i \equiv 0.$$

Since ω_i is real,

$$\operatorname{Re} \sum_{i=1}^{2g} (d\mu_i + \omega_i) v_i \equiv 0.$$

Then, $\tau := \sum (d\mu_i + \omega_i) v_i$ is a closed $\operatorname{Im}(\mathbb{H})$-valued 1-form. And

$$df \wedge \tau = \sum_{i=1}^{2g} df \wedge (d\mu_i + \omega_i) v_i = 0.$$

Thus f is isothermal.

On the contrary, suppose there exists a closed $\operatorname{Im}(\mathbb{H})$-valued 1-form τ satisfying $df \wedge \tau = 0$. Define $A_i := \int_{a_i} \tau$ and $B_i := \int_{b_i} \tau$. Hence, $\tau - \sum_{i=1}^{g} A_i \omega_i - \sum_{i=1}^{g} B_i \omega_{g+i}$ is an exact $\operatorname{Im}(\mathbb{H})$-valued 1-form. It implies there exists $\mu : M \to \operatorname{Im} \mathbb{H}$ such that

$$d\mu = \tau - \sum_{i=1}^{g} A_i \omega_i - \sum_{i=1}^{g} B_i \omega_{g+i}.$$

Then,

$$\begin{aligned} &\mathrm{D}(\mu - \sum_{i=1}^{2g} A_i \mu_i - \sum_{i=1}^{2g} B_i \mu_{g+i}) \\ =& -\frac{df \wedge \left(\tau - \sum_{i=1}^{2g} A_i(\omega_i + d\mu_i) - \sum_{i=1}^{2g} B_i(\omega_{g+i} + d\mu_{g+i})\right)}{|df|^2} \\ =& 0. \end{aligned}$$

Since $\dim(\operatorname{Ker} \mathrm{D}) = 4$, there exist a constant $c \in \mathbb{H}$ such that

$$\mu - \sum_{i=1}^{2g} A_i \mu_i - \sum_{i=1}^{2g} B_i \mu_{g+i} \equiv c.$$

It implies

$$\operatorname{Re}(\sum_{i=1}^{2g} A_i \mu_i + \sum_{i=1}^{2g} B_i \mu_{g+i}) = \operatorname{Re}(\sum_{i=1}^{2g} A_i \mu_i + \sum_{i=1}^{2g} B_i \mu_{g+i} - \mu) \equiv \operatorname{Re}(c).$$

Thus, the 6g+1 functions are linearly dependent over $\mathbb{R}$. $\square$

Instead of $\dot\rho$, we have more general exactness condition on $\dot\lambda$ in terms of geometric quantities, without assuming $\dim(\operatorname{Ker} \mathrm{D}) = 4$. The following theorem has its discrete counterpart in terms of discrete differential forms (Theorem 3.9) and is crucial for the discrete Dirac operator.

Theorem 2.22. *(Vector Valued Schläfli Formula) Suppose an infinitesimal conformal deformation of $f : M \to \mathbb{R}^3$ is given by $\dot\lambda = \frac{1}{2}(\dot u - w)$. Then the 1-form*

$$N \times \big(df \circ (\dot u A + \dot A)\big) - *d\dot u N$$

is exact, where $A : TM \to TM$ is the shape operator of f defined via $dN = df \circ A$.

Proof. In the following, we use Cartan's moving frames. Details for the techniques can be found in Ivey and Landsberg (2003).

For any point $x_0 \in M$, there exists an adapted orthonormal frame (e_1, e_2, e_3) on a neighborhood U of x_0, where $e_3 = N$ is the unit normal. We then lift $f : U \to \mathbb{R}^3$ to $F : U \to ASO(3)$ locally by

$$F(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ f & e_1 & e_2 & e_3 \end{pmatrix} \in ASO(3).$$

Write Mauer Cartan form as

$$\omega = F^{-1}dF = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \omega^1 & 0 & \omega^1_2 & \omega^1_3 \\ \omega^2 & \omega^2_1 & 0 & \omega^2_3 \\ 0 & \omega^3_1 & \omega^3_2 & 0 \end{pmatrix}.$$

From equations $dF = F\omega$ and $\langle e_i, e_j\rangle = \delta_{ij}$, we get

$$\begin{aligned} df &= \omega^1 e_1 + \omega^2 e_2, \\ de_i &= \sum \omega^j_i e_j, \\ \omega^j_i &= -\omega^i_j. \end{aligned}$$

Taking exterior derivative of the first two equations yields

$$\begin{aligned} 0 = d(df) &= (d\omega^1 + \omega^2 \wedge \omega^1_2)e_1 + (d\omega^2 + \omega^1 \wedge \omega^2_1)e_2, \\ 0 = d(de_i) &= \sum_k (d\omega^k_i + \sum_j \omega^j_i \wedge \omega^k_j)e_k. \end{aligned}$$

It implies Cartan's structure equations:

$$\begin{aligned} d\omega^1 &= \omega^1_2 \wedge \omega^2, \\ d\omega^2 &= \omega^2_1 \wedge \omega^1, \\ d\omega^k_i &= \sum_j \omega^k_j \wedge \omega^j_i. \end{aligned}$$

We now consider an infinitesimal conformal deformation given by $\dot\lambda$ and write

$$\dot\lambda = \frac{1}{2}(\dot u - w).$$

Then,

$$\dot F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \dot f & w \times e_1 & w \times e_2 & w \times e_3 \end{pmatrix}.$$

And

$$d\dot F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ d\dot f & dw \times e_1 + w \times de_1 & dw \times e_2 + w \times de_2 & dw \times e_3 + w \times de_3 \end{pmatrix}$$

On the other hand,

$$\begin{aligned} d\dot{F} =& \dot{F}\omega + F\dot{\omega} \\ =& \begin{pmatrix} 0 & 0 & 0 & 0 \\ w\times df & w\times de_1 & w\times de_2 & w\times de_3 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ \dot{\omega}^1 e_1 + \dot{\omega}^2 e_2 & \dot{\omega}_1^2 e_2 + \dot{\omega}_1^3 e_3 & \dot{\omega}_2^1 e_1 + \dot{\omega}_2^3 e_3 & \dot{\omega}_3^1 e_1 + \dot{\omega}_3^2 e_2 \end{pmatrix}. \end{aligned}$$

Since $d\dot{f} = \dot{u}df + w \times df$, comparing two expression of $d\dot{F}$ gives

$$\begin{aligned} \dot{u}df = \dot{u}(\omega^1 e_1 + \omega^2 e_2) &= \dot{\omega}^1 e_1 + \dot{\omega}^2 e_2, \\ dw \times e_1 &= \dot{\omega}_1^2 e_2 + \dot{\omega}_1^3 e_3, \\ dw \times e_2 &= \dot{\omega}_2^1 e_1 + \dot{\omega}_2^3 e_3, \\ dw \times e_3 &= \dot{\omega}_3^1 e_1 + \dot{\omega}_3^2 e_2. \end{aligned}$$

Notice that the shape operator $A : TM \to TM$ is defined via

$$dN = df \circ A.$$

It implies locally,

$$\omega_3^1 e_1 + \omega_3^2 e_2 = (\omega^1 e_1 + \omega^2 e_2) \circ A.$$

Since $\dot{e}_i = w \times e_i$, differentiating both sides,

$$\begin{aligned} \dot{\omega}_3^1 e_1 + \dot{\omega}_3^2 e_2 &= (\dot{\omega}^1 e_1 + \dot{\omega}^2 e_2) \circ A + (\omega^1 e_1 + \omega^2 e_2) \circ \dot{A} \\ &= \dot{u}df \circ A + df \circ \dot{A}. \end{aligned}$$

Hence, the tangential part of dw can be expressed as

$$\begin{aligned} dw^{/\!/} &= e_3 \times (\dot{\omega}_3^1 e_1 + \dot{\omega}_3^2 e_2) \\ &= e_3 \times \big(df \circ (\dot{u}A + \dot{A})\big). \end{aligned}$$

We are left with the normal component of dw. From the structure equations,

$$d\omega^1 = \omega_2^1 \wedge \omega^2.$$

Differentiating both sides,

$$d\dot{u} \wedge \omega^1 + \dot{u}d\omega^1 = d\dot{\omega}^1 = \dot{\omega}_2^1 \wedge \omega^2 + \omega_2^1 \wedge \dot{u}\,\omega^2$$

Since $\dot{u}\,\omega_2^1 \wedge \omega^2 = \dot{u}d\omega^1$,

$$d\dot{u} \wedge \omega^1 = \dot{\omega}_2^1 \wedge \omega^2.$$

Similarly, differentiating the second structure equation $d\omega^2 = \omega_1^2 \wedge \omega^1$ get

$$\begin{aligned} d\dot{u} \wedge \omega^2 &= \dot{\omega}_1^2 \wedge \omega^1 \\ &= -\dot{\omega}_2^1 \wedge \omega^1. \end{aligned}$$

Since $d\dot{u}, \dot{\omega}_1^2 \in span\{\omega^1, \omega^2\}$, the above two equations imply

$$\dot{\omega}_1^2 = -{*}d\dot{u}.$$

Hence,

$$\begin{aligned} \langle dw, e_3 \rangle = \langle dw^\perp, e_3 \rangle &= \langle dw^\perp \times e_1, e_3 \times e_1 \rangle \\ &= \langle \dot{\omega}_1^2 e_2 + \dot{\omega}_1^3 e_3, e_2 \rangle = \dot{\omega}_1^2 = -{*}d\dot{u}. \end{aligned}$$

Thus,

$$dw = N \times \big(df \circ (\dot{u}A + \dot{A})\big) - {*}d\dot{u}N$$

is an exact 1-form. □

Given an exact 1-form and a closed 1-form, we get an exact 2-form via their wedge product. Its integration over the surface thus vanishes. So we have the following corollaries. The first one is the smooth analog of scalar Schläfli Formula.

Corollary 2.23. *Under an infinitesimal conformal deformation,*

$$\int_M \dot{H} + \dot{u}H|df|^2 = 0.$$

PROOF. Suppose an infinitesimal conformal deformation is given by $\dot{\lambda} = \frac{1}{2}(\dot{u} - w)$. Let X be a unit tangent vector.

$$\begin{aligned} df \wedge dw(X, JX) =& df N \wedge df \circ (\dot{u}A + \dot{A}) - *d\dot{u}N)(X, JX) \\ =& - df(JX)df(\dot{u}AJX + \dot{A}JX) - df(X)df(\dot{u}AX + \dot{A}X) \\ &+ df(X)d\dot{u}N + df(JX)d\dot{u}(JX)N \\ =& \langle \dot{u}AJX + \dot{A}JX, JX\rangle + \langle \dot{u}AX + \dot{A}X, X\rangle + \big(\langle \dot{u}AJX\dot{A}JX, X\rangle \\ &- \langle \dot{u}AX + \dot{A}X, JX\rangle\big)N - df(JX)d\dot{u}(X) + df(X)d\dot{u}(JX) \\ =& 2\dot{u}H + 2\dot{H} + df \wedge d\dot{u}(X, JX). \end{aligned}$$

We recover the Dirac operator and the integrability condition.

$$\mathrm{D}\,\dot{\lambda}|df|^2 = -df \wedge d(\frac{\dot{u}}{2} - \frac{w}{2}) = (\dot{u}H + \dot{H})|df|^2.$$

Hence,

$$\int_M \dot{H} + \dot{u}H|df|^2 = \int_M -df \wedge d(\frac{\dot{u}}{2} - \frac{w}{2}) = 0.$$

□

Remark 2.24. *It is an important observation that the quaternionic Dirac operator and the integrability condition is recovered by wedging df with the exact 1-form in vector valued Schläfli formula. This procedure will be taken to get the discrete Dirac operator in Chapter 3.*

Corollary 2.25. *Under an infinitesimal conformal deformation,*

$$\int_M \dot{K} + 2\dot{u}K|df|^2 = 0,$$

which is the infinitesimal version of Gauss Bonnet theorem.

PROOF. Let X be a unit tangent vector.

$$\begin{aligned} Re(dw \wedge dN(X, JX)) =& \mathrm{Re}\,\big((Ndf(\dot{u}AX + \dot{A}X) - *d\dot{u}(X)N)dN(JX) \\ &- (Ndf(\dot{u}AJX + \dot{A}JX) - *d\dot{u}(JX)N)dN(X)\big) \\ =& - \langle \dot{u}JAX + J\dot{A}X, AJX\rangle + \langle \dot{u}JAJX + J\dot{A}JX, AX\rangle \\ =& - \dot{u}(\langle JAX, AJX\rangle + \langle AJX, JAX\rangle) \\ &- (\langle J\dot{A}X, AJX\rangle + \langle \dot{A}JX, JAX\rangle) \\ =& - 2\dot{u}K - \dot{K}. \end{aligned}$$

Hence,

$$\int_M \dot{K} + 2\dot{u}K|df|^2 = -Re(\int_M dw \wedge df) = 0.$$

□

6. Vector Analysis of the Quaternionic Dirac Operator

We decompose a given $\lambda : M \to \mathbb{H}$ into the form $\lambda = g + df(Y) + hN$, where g, h are real-valued functions, Y is a tangent vector field and N is the normal vector field on M. We want to study $\mathrm{D}\,\lambda$ under this decomposition.

Take a unit tangent vector X on M in the following. We first consider the action of the Dirac operator on the scalar component.

$$\begin{aligned}-df \wedge dg(X, JX) &= -df(X)dg(JX) + df(JX)dg(X)\\ &= Ndf(dg(X)X + dg(JX)JX)\\ &= Ndf(\operatorname{grad} g).\end{aligned}$$

Then we consider the normal component.

$$\begin{aligned}-df \wedge d(hN)(X, JX) &= \big((-df \wedge dh)N - hdf \wedge dN\big)(X, JX)\\ &= Ndf(\operatorname{grad} h)N - h\big(df(X)dN(JX) - df(JX)dN(X)\big)\\ &= df(\operatorname{grad} h) - 2hHN.\end{aligned}$$

Finally we look at the tangential component. Notice that for an immersed surface in Euclidean space, the induced Levi-Civita connection is as follows: for any tangent vector field Y and tangent vector Z,

$$\begin{aligned}df(\nabla_Z Y) &= d(df(Y))(Z) - \langle d(df(Y))(Z), N\rangle N\\ &= d(df(Y))(Z) + \langle df(Y), df(AZ)\rangle N\\ &= d(df(Y))(Z) + \langle Y, AZ\rangle N\end{aligned}$$

where A is the self-adjoint shape operator of the immersion f. And we recall the definition of curl and divergent operator of a tangent vector field Y:

$$\begin{aligned}\operatorname{div}(Y) :&= \langle X, \nabla_X Y\rangle + \langle X, \nabla_X Y\rangle,\\ \operatorname{curl}(Y) :&= \langle JX, \nabla_X Y\rangle - \langle X, \nabla_{JX} Y\rangle\\ &= -\langle X, \nabla_X JY\rangle - \langle JX, \nabla_{JX} JY\rangle\\ &= -\operatorname{div}(JY).\end{aligned}$$

Then,

$$\begin{aligned}-df \wedge d(df(Y))(X, JX) =& -df(X)\big(df(\nabla_{JX} Y) - \langle Y, AJX\rangle N\big)\\ &+ df(JX)\big(df(\nabla_X Y) - \langle Y, AX\rangle N\big)\\ =&\langle X, \nabla_{JX} Y\rangle - \langle JX, \nabla_X Y\rangle - \langle JX, \nabla_{JX} Y\rangle N\\ &+ \langle -X, \nabla_X Y\rangle N - \langle AY, JX\rangle df(JX) - \langle AY, X\rangle df(X)\\ =& -\operatorname{curl} Y - (\operatorname{div} Y)N - df(\mathrm{A}\,Y).\end{aligned}$$

The above calculation is concluded with the following theorem.

Theorem 2.26. *Given an immersion $f : M \to \mathbb{R}^3$ and a $\mathbb{H}$-valued function $\lambda = g + df(Y) + hN$, where g,$df(Y)$ and hN are its scalar, tangential and normal components. Then*

$$\mathrm{D}\,\lambda = -\operatorname{curl} Y + df(J \operatorname{grad} g - \mathrm{A}\,Y + \operatorname{grad} h) - \big((\operatorname{div} Y) + 2hH\big)N.$$

From the above formula, we can express Laplace operator in terms of the quaternion Dirac operator D. This expression has its analogue in the discrete theory (Theorem 3.33).

Corollary 2.27. *For any real valued function $g : M \to \mathbb{R}$,*

$$\operatorname{Re}(\mathrm{D}^2\, g) = -\operatorname{div}(\operatorname{grad} g) =: \Delta g.$$

We can write the formula of the discrete Dirac operator into a matrix form.

$$\mathrm{D}\begin{pmatrix} g \\ Y \\ hN \end{pmatrix} = \begin{pmatrix} 0 & -\operatorname{curl} & 0 \\ J\operatorname{grad} & -\mathrm{A} & \operatorname{grad} \\ 0 & -\operatorname{div} & -2H \end{pmatrix} \begin{pmatrix} g \\ Y \\ hN \end{pmatrix}.$$

And so we have formula for D^2.

Corollary 2.28.

$$\mathrm{D}^2 = \begin{pmatrix} \Delta & curl \circ A & 0 \\ -\mathrm{A}\,J\operatorname{grad} & -J\operatorname{grad}\circ\operatorname{curl} + A^2 - \operatorname{grad}\circ\operatorname{div} & -\mathrm{A}\operatorname{grad} - 2\operatorname{grad}\circ H \\ 0 & \operatorname{div}\circ\mathrm{A} + 2H\operatorname{div} & \Delta + 4H^2 \end{pmatrix}.$$

CHAPTER 3

Conformal Deformations of Triangulated Surfaces

In the previous chapter, it was shown that two immersions related under a spin transformation are conformally equivalent. In order to discuss the discretized case, we need a notion of conformal equivalence of triangulated surfaces. In section 1, conformal equivalence of triangular meshes is reviewed.

With this notation, we consider infinitesimal conformal deformations of triangulated surfaces, by prescribing changes to triangles in sections 2 and 3. These results are collected from the blog "Discrete Spin Geometry".

In sections 4 and 5, the discrete analogues of the results from the previous chapter are derived and shown to have nice properties. The discrete theory also establishes a connection with the classical result—the Schläfli Formula.

1. Conformal Equivalence of Triangulated Surfaces

This section reviews the definition of the conformal equivalence of triangulated surfaces introduced in Luo (2004) and applied to conformal parametrization (Springborn et al., 2008). It enjoys important properties as in the smooth case. Given an immersion into Euclidean space, a conformal structure is induced on the abstract triangulated surface. A Möbius transformation of a given immersion produces a new immersion conformally equivalent to the original one. Further study of this conformal equivalence notion and its connection to complete hyperbolic metric on punctured surfaces and circle packing theory are discussed in Bobenko et al. (2010).

Definition 3.1. *A discrete metric on a triangulated surface M is a length function $l : E \to \mathbb{R}_{>0}$ such that the triangle inequality is satisfied for all triangles $(ijk) \in T$.*

Definition 3.2. *An (almost) immersion of a triangulated surface M into $\mathbb{R}^3$ is a continuous map $f : M \to \mathbb{R}^3$ such that*

(1) *f is piecewise linear on each triangle;*
(2) *the image of each triangle of M under f is a non-degenerate triangle in Euclidean space and*
(3) *the images of any two neighboring triangles intersect only at their common edge.*

Remark 3.3. *From the definition, an immersion of a triangulated surface M is uniquely determined by its values on vertices. And dihedral angles $\alpha \in (-\pi, \pi)$ are well defined on edges.*

Given an immersion f of a triangulated surface M into $\mathbb{R}^3$, a discrete metric l is induced on M by

$$\forall e_{ij} \in E, \quad l_{ij} = |f_j - f_i|.$$

Definition 3.4. *(Luo, 2004) Two discrete metric l and $\tilde{l}$ on M are discretely conformally equivalent if there exists $u : V \to \mathbb{R}$, assigning numbers u_i to vertices v_i, such that*

$$\tilde{l}_{ij} = e^{\frac{u_i+u_j}{2}} l_{ij}. \tag{3.1}$$

Two immersions f and $\tilde{f}$ are conformally equivalent if their induced discrete metrics are conformally equivalent.

Remark 3.5. *The defined notion indeed satisfies the conditions for being an equivalence relation, i.e. reflexivity, symmetry and transitivity. It defines equivalence class on discrete metric.*

Theorem 3.6. *Suppose $f : M \to \mathbb{R}^3$ is an immersion of a triangulated surface M. Then for any Möbius transformation ϕ of $\mathbb{R}^3$, the two immersions f and $\phi \circ f$ are discretely conformally equivalent.*

Proof. Möbius transformations can be generated by translation, scaling and inversion under the unit sphere $\mathbb{S}^2$. The conformal equivalence holds obviously under scaling and translation. By transitivity of the equivalence relation, it suffices to show the case where ϕ is the inversion under the unit sphere $\mathbb{S}^2$, which follows from

$$\begin{aligned} \left|\left|\frac{f_i}{|f_i|^2} - \frac{f_j}{|f_j|^2}\right|\right|^2 &= \frac{1}{|f_i|^2} + \frac{1}{|f_j|^2} - \frac{2}{|f_i|^2|f_j|^2}\langle f_i, f_j\rangle \\ &= \frac{1}{|f_i|^2|f_j|^2}||f_i - f_j||^2. \end{aligned}$$

□

To prepare for the following sections, we make a remark.

Remark 3.7. *Given an immersion of a triangulated surface into $\mathbb{R}^3$, we want to find conformally equivalent surfaces with prescribed mean curvature half-density, as analogous to the smooth theory. To avoid inputting more variables than the degree of freedom of available immersions, we need a clue for the domain to define the mean curvature half-density, by heuristic counting on the dimension of freedom.*

For a triangulated surface in $\mathbb{R}^3$ of genus g with V vertices, the degree of freedom is $3V - 7$ up to rigid transformations and scaling. Each length function is in some conformal class. By the definition of discrete conformal equivalence, the dimension of conformal class is $E - V = 2V - 6 + 6g$. The dimension of the space of immersed triangulated surfaces in a given conformal class is $(3V - 7) - (E - V) = V - 1 - 6g$. In particular, for surfaces with genus $g = 0$, we have $(3V - 7) - (E - V) = V - 1$.

2. Infinitesimal Conformal Deformations and the Discrete Dirac Operator

We are going to look at infinitesimal conformal deformations of triangulated surfaces and derive the discrete Dirac operator, as an analogue in the smooth case given by $\dot{\lambda}$ under the integrability condition: $D\dot{\lambda} = \dot{\rho}$. In the smooth theory, by considering Cartan's moving frames, we get an exact 1-form (Theorem 2.22). By wedging it with df, we again get the integrability condition and the quaternionic Dirac operator (Remark 2.24). We proceed the same way for triangulated surfaces. By prescribing changes to triangles, we get a co-exact discrete 1-form (Theorem 3.9). And we have integrability condition in terms of the discrete Dirac operator by wedging it with df (Lemma 3.16).

On a triangulated surface f, we have frames $(T, N, T \times N)$ on oriented edges. Given a $\mathbb{H}$-valued function λ defined on oriented edges, new frames $(\tilde{T}, \tilde{N}, \tilde{T} \times \tilde{N})$ can be obtained by multiplying with quaternions on oriented edges $e \subset \phi \in F$:

$$(\tilde{T}_e, \tilde{N}_\phi, \tilde{T}_e \times \tilde{N}_e) = \lambda_e^{-1}(T_e, N_\phi, T_e \times N_\phi)\lambda_e.$$

For these frames coming from a new discrete surface $\tilde{f}$ determined by

$$d\tilde{f} = \overline{\lambda} df \lambda,$$

the quaternion function λ necessarily satisfies three conditions:

(1) For any pair of oriented edges $(e,-e)$, the stretch-rotations given by $y \mapsto \overline{\lambda}_{\pm e} y \lambda_{\pm e}$ take $df(e)$ to the same vector $d\tilde{f}(e)$.
(2) For all oriented edges in the same face ϕ, the rotation $y \mapsto \lambda_e^{-1} y \lambda_e$ takes the face normal N_ϕ to the same unit vector.
(3) For any face ϕ consists of oriented edges (e_1, e_2, e_3), the closedness condition $0 = \overline{\lambda}_{e_1} df(e_1)\lambda_{e_1} + \overline{\lambda}_{e_2} df(e_2)\lambda_{e_2} + \overline{\lambda}_{e_3} df(e_3)\lambda_{e_3}$ is satisfied.

Now we consider the three conditions under infinitesimal conformal transformations. Suppose a 1-parameter family of spin transformations of f is given by $\lambda(t)$ with $\lambda(0) = 1$. We are going to differentiate the equations given in the conditions with respect to t and evaluate them at $t = 0$.

We first use an Ansatz to uniquely split $\dot{\lambda}_e$ into scalar, tangential and normal components

$$\dot{\lambda}_e := \frac{\sigma_e}{2} - \frac{\omega_e}{2} N_\phi - \frac{Y_e}{2}. \tag{3.2}$$

Consider the change of the normal vector N_ϕ, where $e \subset \phi \in F$,

$$\begin{aligned}
(\lambda_e^{-1} \dot{N_\phi \lambda_e}) &= \dot{\Big(\frac{\overline{\lambda_e} N_\phi \lambda_e}{|\lambda_e|^2}\Big)} \\
&= \overline{\dot{\lambda}_e} N_\phi + N_\phi \dot{\lambda}_e - (\dot{\lambda}_e + \overline{\dot{\lambda}_e}) N_\phi \\
&= N_\phi \dot{\lambda}_e - \dot{\lambda}_e N_\phi \\
&= Y_e \times N_\phi.
\end{aligned}$$

Thus, the condition (2) implies for any face ϕ consists of oriented edges (e_1, e_2, e_3),

$$Y_\phi := Y_{e_1} = Y_{e_2} = Y_{e_3}.$$

From condition (3), we have

$$\mathrm{Im}\left(df(e_1)\dot{\lambda}_{e_1} + df(e_2)\dot{\lambda}_{e_2} + df(e_3)\dot{\lambda}_{e_3}\right) = 0.$$

Notice that

$$\mathrm{Re}(df(e_1)\dot{\lambda}_{e_1} + df(e_2)\dot{\lambda}_{e_2} + df(e_3)\dot{\lambda}_{e_3}) = \langle df(e_1) + df(e_2) + df(e_3), \frac{Y_\phi}{2} \rangle = 0.$$

Hence,

$$df(e_1)\dot{\lambda}_{e_1} + df(e_2)\dot{\lambda}_{e_2} + df(e_3)\dot{\lambda}_{e_3} = 0.$$

By substituting Ansatz (3.2),

$$df(e_1)(\sigma_{e_1} - \omega_{e_1} N_\phi) + df(e_2)(\sigma_{e_2} - \omega_{e_2} N_\phi) + df(e_3)(\sigma_{e_3} - \omega_{e_3} N_\phi) = 0. \tag{3.3}$$

Note that we have the following equalities for $e_1, e_2, e_3 \subset \phi$,

$$\begin{aligned}
0 &= \langle N_\phi df(e_i), df(e_i) \rangle, \\
2\,\mathrm{Area}(\phi) &= \langle N_\phi df(e_i), df(e_{i+1}) \rangle, \\
\langle N_\phi df(e_i), N_\phi df(e_i) \rangle &= \langle df(e_i), df(e_i) \rangle.
\end{aligned}$$

Since $df(e_1) \in span\{N_\phi df(e_2), N_\phi df(e_3)\}$, from the above equalities

$$\begin{aligned}
df(e_1) &= \frac{\langle df(e_1), df(e_3) \rangle}{\langle N_\phi df(e_2), df(e_3) \rangle} N_\phi df(e_2) + \frac{\langle df(e_1), df(e_2) \rangle}{\langle N_\phi df(e_3), df(e_2) \rangle} N_\phi df(e_3) \\
&= \frac{\langle df(e_1), df(e_3) \rangle}{\langle N_\phi df(e_3), df(e_1) \rangle} N_\phi df(e_2) + \frac{\langle df(e_1), df(e_2) \rangle}{\langle N_\phi df(e_2), df(e_1) \rangle} N_\phi df(e_3) \\
&= \cot(\beta_3) N_\phi df(e_3) - \cot(\beta_2) N_\phi df(e_2).
\end{aligned}$$

Similarly, by cyclic permutation, we also have

$$df(e_2) = cot(\beta_1) N_\phi df(e_1) - cot(\beta_3) N_\phi df(e_3),$$

$$df(e_3) = cot(\beta_2)N_\phi df(e_2) - cot(\beta_1)N_\phi df(e_1).$$

Substituting them into (3.3),

$$\begin{aligned} 0 = \quad & \sigma_1\big(\cot(\beta_3)N_\phi df(e_3) - \cot(\beta_2)N_\phi df(e_2)\big) + \omega_1 N_\phi df(e_1) \\ & + \sigma_2\big(cot(\beta_1)N_\phi df(e_1) - cot(\beta_3)N_\phi df(e_3)\big) + \omega_2 N_\phi df(e_2) \\ & + \sigma_3\big(cot(\beta_2)N_\phi df(e_2) - cot(\beta_1)N_\phi df(e_1)\big) + \omega_3 N_\phi df(e_3) \\ = \quad & \big(\omega_1 + \cot\beta_1(\sigma_2 - \sigma_3)\big)N_\phi df(\mathbf{e}_1) \\ & + \big(\omega_2 + \cot\beta_2(\sigma_3 - \sigma_1)\big)N_\phi df(\mathbf{e}_2) \\ & + \big(\omega_3 + \cot\beta_3(\sigma_1 - \sigma_2)\big)N_\phi df(\mathbf{e}_3). \end{aligned}$$

Since $N_\phi df(e_1)$,$N_\phi df(e_2)$ and $N_\phi df(e_3)$ span an affine plane and

$$N_\phi df(e_1) + N_\phi df(e_2) + N_\phi df(e_3) = 0,$$

there exists unique ω_ϕ such that

$$\omega_\phi = \omega_1 + \cot\beta_1(\sigma_2 - \sigma_3) = \omega_2 + \cot\beta_2(\sigma_3 - \sigma_1) = \omega_3 + \cot\beta_3(\sigma_1 - \sigma_2).$$

Rewriting the formula, we get

$$\begin{aligned} \omega_{e_1} &= \omega_\varphi - \cot\beta_1(\sigma_{e_2} - \sigma_{e_3}), \\ \omega_{e_2} &= \omega_\varphi - \cot\beta_2(\sigma_{e_3} - \sigma_{e_1}), \\ \omega_{e_3} &= \omega_\varphi - \cot\beta_3(\sigma_{e_1} - \sigma_{e_2}), \end{aligned}$$

where β_i is the angle of the triangle ϕ at the vertex i opposite to the edge e_i.

For a conformal deformation, by the definition of the conformal equivalence, the scaling is the average of some real-valued function u defined on vertices, which is uniquely determined by σ on edges,

$$\begin{aligned} \sigma_{e_1} &= \frac{\dot{u}_2 + \dot{u}_3}{2}, \\ \sigma_{e_2} &= \frac{\dot{u}_3 + \dot{u}_1}{2}, \\ \sigma_{e_3} &= \frac{\dot{u}_1 + \dot{u}_2}{2}. \end{aligned}$$

So,

$$\begin{aligned} \omega_{e_1} &= \omega_\varphi - \frac{\cot\beta_1}{2}(\dot{u}_3 - \dot{u}_2), \\ \omega_{e_2} &= \omega_\varphi - \frac{\cot\beta_2}{2}(\dot{u}_1 - \dot{u}_3), \\ \omega_{e_3} &= \omega_\varphi - \frac{\cot\beta_3}{2}(\dot{u}_2 - \dot{u}_1). \end{aligned}$$

Hence, the conditions (2) and (3) imply under a conformal deformation, there exist a unique scalar function $\dot{u}_i$ defined on vertices, ω_ϕ on faces and a vector valued function $Y_\phi \in \mathbb{R}^3$ with $Y_\phi \perp N_\phi$ defined on faces such that

$$\dot{\lambda}_{e_i} = \frac{\dot{u}_j + \dot{u}_k}{4} - \frac{1}{2}(\omega_\varphi - \frac{\cot\beta_i}{2}(\dot{u}_k - \dot{u}_j))N_\varphi - \frac{Y_\varphi}{2}.$$

We then define on each face and each vertex new variables

$$\begin{aligned} Z_\varphi &= -(\frac{\omega_\varphi N_\varphi}{2} + \frac{Y_\varphi}{2}) \in \mathbb{R}^3, \\ u_i &= \frac{\dot{u}_i}{2}. \end{aligned}$$

Rewrite $\dot\lambda_{e_i}$ as

$$\dot\lambda_{e_i} = \frac{u_j + u_k}{2} + Z_\varphi + \frac{\cot\beta_i}{2}(u_k - u_j)N_\varphi.$$

We now study the first condition. Differentiating the equation given in the condition (1) yields

$$\begin{aligned} \overline{\dot\lambda}_e df(e) + df(e)\dot\lambda_e &= \overline{\dot\lambda}_{-e} df(e) + df(e)\dot\lambda_{-e} \\ \mathrm{Im}\left(df(e)\dot\lambda_e\right) &= \mathrm{Im}\left(df(e)\dot\lambda_{-e}\right) \\ \mathrm{Re}(\dot\lambda_e)df(e) + df(e)\times\mathrm{Im}(\dot\lambda_e) &= \mathrm{Re}(\dot\lambda_{-e})df(e) + df(e)\times\mathrm{Im}(\dot\lambda_{-e}) \end{aligned}$$

It implies

$$\begin{aligned} \mathrm{Re}(\dot\lambda_e) &= \mathrm{Re}(\dot\lambda_{-e}) \\ \mathrm{Im}(\dot\lambda_e) - \mathrm{Im}(\dot\lambda_{-e}) &/\!/ df(e) \end{aligned}$$

Hence, on every oriented edge e, there exists unique real number $\dot\alpha_e$ with $\dot\alpha_e = \dot\alpha_{-e}$ such that

$$\dot\lambda_{-e} = \dot\lambda_e - \frac{\dot\alpha_e}{2}T_e.$$

Lemma 3.8. *If the immersed surface deforms according to $\dot\lambda$, then $\dot\alpha_e$ in the above equation is the change of the dihedral angle at the edge e.*

PROOF. Suppose $e_{ij} \in E$ and $(ijk), (ji\tilde k) \in F$ are neighboring triangles. Denote α as the dihedral angle on e_{ij}.

$$\sin\alpha = \langle N_{ijk} \times N_{ji\tilde k}, T_{e_{ij}}\rangle$$

Differentiating both sides,

$$\begin{aligned} \dot\alpha\cos\alpha =& \langle \dot N_{ijk} \times N_{ji\tilde k} + N_{ijk} \times \dot N_{ji\tilde k}, T_{e_{ij}}\rangle \\ =& \langle \left(N_{ijk} \times \mathrm{Im}(2\dot\lambda_{e_{ij}})\right) \times N_{ji\tilde k} + N_{ijk} \times \left(N_{ji\tilde k} \times \mathrm{Im}(2\dot\lambda_{e_{ji}})\right), T_{e_{ij}}\rangle \\ =& \big\langle\, \mathrm{Im}(2\dot\lambda_{e_{ij}})\langle N_{ji\tilde k}, N_{ijk}\rangle - N_{ijk}\langle N_{ji\tilde k}, \mathrm{Im}(2\dot\lambda_{e_{ij}})\rangle \\ &+ N_{ij\tilde k}\langle N_{jik}, \mathrm{Im}(2\dot\lambda_{e_{ji}})\rangle - \mathrm{Im}(2\dot\lambda_{e_{ji}})\langle N_{ji\tilde k}, N_{ijk}\rangle, T_{e_{ij}}\big\rangle \\ =& \cos\alpha\langle \mathrm{Im}(2\dot\lambda_{e_{ij}}) - \mathrm{Im}(2\dot\lambda_{e_{ji}}), T_{e_{ij}}\rangle. \end{aligned}$$

Since we know

$$\begin{aligned} \mathrm{Re}(\dot\lambda_e) &= \mathrm{Re}(\dot\lambda_{-e}), \\ \mathrm{Im}(\dot\lambda_e) - \mathrm{Im}(\dot\lambda_{-e}) &/\!/ df(e), \end{aligned}$$

we have

$$\dot\lambda_{e_{ij}} - \dot\lambda_{e_{ji}} = \frac{\dot\alpha}{2}T_{e_{ij}}.$$

□

The condition (1) then becomes

$$Z_{\tilde\varphi_{ji\tilde k}} - Z_{\varphi_{ijk}} = (u_j - u_i)\left(\frac{\cot\beta_k}{2}N_{\varphi_{ijk}} + \frac{\cot\beta_{\tilde k}}{2}N_{\varphi_{ji\tilde k}}\right) - \frac{\dot\alpha_{e_{ij}}}{2}T_{e_{ij}}. \tag{3.4}$$

This formula is analogous to the vector-valued Schläfli formula in the smooth theory (Theorem 2.22). We first formulate it here as a theorem and study this co-exact 1-form further in Section 4.

Theorem 3.9. *Suppose an infinitesimal conformal deformation of an immersed triangulated surface M is given by $(u, Z) \in \mathbb{R}^{V+3F}$. Then, the discrete 1-form on each edge e_{ij} given by*

$$du(e_{ij})(\frac{\cot \beta_k}{2} N_{\varphi_{ijk}} + \frac{\cot \beta_{\tilde{k}}}{2} N_{\varphi_{i\tilde{k}j}}) - \frac{\dot{\alpha}_{e_{ij}}}{2} T_{e_{ij}}$$

is co-exact, where β_i is the interior angle of triangle (ijk) at vertex i and $\dot{\alpha}_{e_{ij}}$ is the change of dihedral angle on edge (ij).

We proceed to derive the discrete Dirac operator by multiplying both sides by df as in the smooth case (Remark 2.24) . In the following, we write $dZ(*e) := Z(left(e)) - Z(right(e))$ and $df(*e) := (\frac{\cot \beta}{2} N_\varphi + \frac{\cot \tilde{\beta}}{2} N_{\tilde{\varphi}}) df(e)$, which is the vector from circumcenter of the right face of the edge e to that of the left face under the immersion f. We rewrite Equation (3.4) as

$$-dZ(*e_{ij}) = du(e_{ij})(\frac{\cot \beta_k}{2} N_{\varphi_{ijk}} + \frac{\cot \beta_{\tilde{k}}}{2} N_{\varphi_{ji\tilde{k}}}) - \frac{\dot{\alpha}_{e_{ij}}}{2} T_{e_{ij}}.$$

We multiply both sides by $df(e_{ij})$ via an inner product and a cross product

$$\langle df(e), dZ(*e) \rangle = \frac{\dot{\alpha}_e}{2} |df(e)|, \tag{3.5}$$

$$-df(e) \times dZ(*e) + df(*e) du(e) = 0. \tag{3.6}$$

The following theorem concludes the above derivation.

Theorem 3.10. *Denote $\mathbb{R}^{V+3F}$ as the space of pairs (u, Z), where u is a real valued function defined on vertices and Z is a $\mathbb{R}^3$-valued function on faces. Suppose an infinitesimal conformal deformation of $f : M \to \mathbb{R}^3$ is given by $\dot{f}$. Then, there exists unique $(u, Z) \in \mathbb{R}^{V+3F}$ such that for any face $\phi = (ijk)$*

$$\dot{\lambda}_{e_{jk}} = \frac{u_j + u_k}{2} + Z_\phi + \frac{\cot \beta_i}{2}(u_k - u_j) N_\phi. \tag{3.7}$$

and for any edge $e \in \tilde{E}$

$$-df(e) \times dZ(*e) + df(*e) du(e) = 0. \tag{3.8}$$

It relates to $\dot{f}$ via

$$d\dot{f}(e) = 2\,\mathrm{Im}(df(e)\dot{\lambda}_e) = 2\,\mathrm{Im}(df(e)\dot{\lambda}_{-e}).$$

Motivated by Remark 3.7, we define $\dot{\rho} : V \to \mathbb{R}$ as

$$\dot{\rho}_i := \sum_{ij \in E:i} \langle df(e_{ij}), dZ(*e_{ij}) \rangle.$$

We then further have

$$\dot{\rho}_i = \sum_{ij \in E:i} \frac{\dot{\alpha}_{e_{ij}}}{2} |df(e_{ij})|.$$

Corollary 3.11. *Suppose $f : M \to \mathbb{R}^3$ is an immersion of a simply connected closed triangulated surface. Given $(u, Z) \in \mathbb{R}^{V+3F}$ satisfying (3.8) and $\dot{\lambda}$ in the form of (3.7), then there exists an infinitesimal conformal deformation of f, $\dot{f} : M \to \mathbb{R}^3$ such that*

$$d\dot{f} = 2\,\mathrm{Im}(df(e)\dot{\lambda}_e)$$

Proof. Note that since $(u, Z) \in \mathbb{R}^{V+3F}$ satisfying (3.8) implies

$$\mathrm{Im}\left(df(e)\dot{\lambda}_e\right) = \mathrm{Im}\left(df(e)\dot{\lambda}_{-e}\right).$$

Hence, the change of edge vector $df(e)$ is well defined by $\dot\lambda$. And $\dot\lambda$ is of the form (3.7) implies

$$\operatorname{Im}\left(df(e_1)\dot\lambda_{e_1} + df(e_2)\dot\lambda_{e_2} + df(e_3)\dot\lambda_{e_3}\right) = 0.$$

So, $2\operatorname{Im}(df(e)\dot\lambda_e)$ is a closed discrete 1-form. For genus 0 surface, a closed 1-form is exact. Thus, there exists an infinitesimal conformal deformation of f, $\dot f : M \to \mathbb{R}^3$ such that

$$d\dot f = 2\operatorname{Im}(df(e)\dot\lambda_e),$$

which is also conformal since

$$\operatorname{Re}\left(\dot\lambda_{e_{jk}}\right) = \frac{u_j + u_k}{2}.$$

□

Remark 3.12. *In the above theorem, $\dot\lambda$ given by the formula* (3.7) *should be regarded as having its real part u on vertices and its imaginary part Z on faces.*

In order to define the discrete Dirac operator, we need some notations.

Definition 3.13. *Denote $\mathbb{R}^{V+2E}$ as the space of pairs (α, W), where α is a real valued function defined on vertices and W is $\mathbb{R}^3$-valued function on (unoriented) edges such that on every edge e, W_e is perpendicular to $df(e)$.*

For a general pair $(u, Z) \in \mathbb{R}^{V+3F}$, as before,

$$\dot\rho_i := \sum_{ij\in E} \langle df(e_{ij}), dZ(*e_{ij})\rangle$$

is a scalar valued discrete 2-form on vertices and

$$U_e := -df(e) \times dZ(*e) + df(*e)du(e)$$

is a vector valued function on edges, which are regarded as discrete 2-forms on edges.

Given a 2 form,we can define a functional on functions by multiplying them together and then take the integration over the surface.

Definition 3.14. *$(\dot\rho, U)$ in above formula define a linear functional on $\mathbb{R}^{V+2E}$ by*

$$((\dot\rho, U), (\alpha, W))_{\mathbb{H}} = \sum_i \dot\rho_i\alpha_i + \sum_{ij}\langle U_{ij}, W_{ij}\rangle \qquad \forall(\alpha, W) \in \mathbb{R}^{V+2E}.$$

Hence, $(\dot\rho, U)$ can be identified as an element in $(\mathbb{R}^{V+2E})^*$.

We now define the quaternion Dirac operator in the discrete analogue.

Definition 3.15. *The discrete Dirac operator is a linear map $\mathrm{D} : \mathbb{R}^{V+3F} \to (\mathbb{R}^{V+2E})^*$, $(u, Z) \mapsto (\dot\rho, U)$ where*

$$\dot\rho_i = \sum_{ij\in E:i} \langle df(e_{ij}), dZ(*e_{ij})\rangle,$$
$$U_e = -df(e) \times dZ(*e) + df(*e)du(e).$$

In particular, the image of D *is regarded as a $\mathbb{H}$-valued function with its real part* $\operatorname{Re}(\mathrm{D}) = \dot\rho$ *on vertices and imaginary part* $\operatorname{Im}(\mathrm{D}) = U$ *on edges.*

We rewrite the necessary condition (3.8) for conformal deformations in terms of D from Theorem (3.10).

Lemma 3.16. *If $(u, Z) \in \mathbb{R}^{V+3F}$ gives an infinitesimal conformal deformation of $f : M \to \mathbb{R}^3$, it satisfies*

$$\operatorname{Im}\mathrm{D}(u, Z) = 0.$$

Remark 3.17. *The pairing $(\ ,\ )_{\mathbb{H}}$ is motivated by the formula*

$$\langle\lambda_1, \lambda_2\rangle_{\mathbb{H}} = \operatorname{Re}(\lambda_1)\operatorname{Re}(\lambda_2) + \langle\operatorname{Im}(\lambda_1), \operatorname{Im}(\lambda_2)\rangle_{\mathbb{R}^3}.$$

Remark 3.18. *The image of* D *should be identified as in* $(\mathbb{R}^{V+2E})^*$. *Given two smooth real functions* λ_1, λ_2 *on* M, *an area 2-form* dA *is necessary in order to define their inner product by* $\int_M \lambda_1\lambda_2\, dA$. *The same should hold on triangulated surfaces. Area elements at vertices and edges are necessary to weight the products in* $\mathbb{R}^{V+2E}$. *Since the image of* D *is a discrete 2-form, it already incorporate area elements. Hence it can be naturally paired up with elements in* $\mathbb{R}^{V+2E}$ *without weighted by area elements, thus defining an element in* $(\mathbb{R}^{V+2E})^*$.

We look at a consequence of the above derivation.

Corollary 3.19. *(Scalar Schläfli Formula) Suppose a conformal immersion* $f : M \to \mathbb{R}^3$ *is under an infinitesimal conformal deformation, then the changes of dihedral angles satisfies*

$$\sum_{i \in V} \dot\rho_i = \sum_{e \in E} \dot\alpha_e |df(e)| = 0.$$

Proof. Since the infinitesimal conformal deformation is given by some $\dot\lambda$, its corresponding (u, Z) satisfies

$$\begin{aligned}\sum_{ij \in E:i} \frac{\dot\alpha_{ij}}{2}|df(e_{ij})| &= (\operatorname{Re} \mathrm{D}(u, Z))_i \\ &= \sum_{ij\in E:i} \langle df(e_{ij}), dZ(*e_{ij})\rangle.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{ij \in E} \dot\alpha_{ij}|df(e_{ij})| &= \sum_i \sum_{ij \in E:i} \frac{\dot\alpha_{ij}}{2}|df(e_{ij})| \\ &= \sum_{ij\in E} \langle df(e_{ij}), dZ(*e_{ij})\rangle.\end{aligned}$$

Notice that since the summand in the right hand side is an exact discrete 2-form, the sum is zero. We get

$$\sum_e \dot\alpha_e |df(e)| = 0.$$

□

Remark 3.20. *Scalar Schläfli Formula holds for general deformations. (See Pak (2010)) One can notice that the crucial point is* $\frac{\dot\alpha_{ij}}{2}|df(e_{ij})| = \langle df(e_{ij}), dZ(*e_{ij})\rangle$. *Observing the derivation of such condition, it does not involve any conformality assumption.*

As in the smooth theory, we are interested in the dimension of Ker(D). An observation from the definition of D leads to the following.

Corollary 3.21. *The kernel of* D *contains constant functions* $(u, Z) \in \mathbb{R}^{V+3F}$ *where* u *is constant on all vertices and* Z *is constant on all faces. Hence,*

$$\dim\big(\operatorname{Ker}(\mathrm{D})\big) \geq 4.$$

3. Adjoint of the Discrete Dirac Operator

The quaternionic Dirac operator in the smooth theory is self-adjoint. Although the domain and the target space of the discrete Dirac operator $\mathrm{D} : \mathbb{R}^{V+3F} \to (\mathbb{R}^{V+2E})^*$ are different and so not self-adjoint, D and its adjoint still share nice properties and consequences as in the smooth case. We are now going to derive the adjoint of D, $\mathrm{D}^* : \mathbb{R}^{V+2E} \to (\mathbb{R}^{V+3F})^*$, with respect to $(\ ,\ )_{\mathbb{H}}$

In the following, let $(u,Z) \in \mathbb{R}^{V+3F}$ and $(\alpha, W) \in \mathbb{R}^{V+2E}$. By definition, D^* satisfies

$$\big((u,z), \mathrm{D}^*(\alpha, W)\big)_{\mathbb{H}} = \big(\mathrm{D}(u,z), (\alpha, W)\big)_{\mathbb{H}}.$$

In particular, consider the following,

$$\begin{aligned}
((u,z), \mathrm{D}^*(\alpha,0))_{\mathbb{H}} =& (\mathrm{D}(u,z), (\alpha,0))_{\mathbb{H}} \\
=& \sum_i \alpha_i \sum_{ij\in E} \langle df(e_{ij}), dZ(*e_{ij})\rangle \\
=& \sum_{ijk\in F} \langle \alpha_i df(e_{ij}) + \alpha_j df(e_{jk}) + \alpha_k df(e_{ki}), Z_{ijk}\rangle \\
& - \sum_{ijk\in F} \langle \alpha_j df(e_{ji}) + \alpha_k df(e_{kj}) + \alpha_i df(e_{ik}), Z_{ijk}\rangle \\
=& - \sum_{ijk\in F} \langle \alpha_i df(e_{jk}) + \alpha_j df(e_{ki}) + \alpha_k df(e_{ij}), Z_{ijk}\rangle, \\
((u,0), \mathrm{D}^*(0,W))_{\mathbb{H}} =& (\mathrm{D}(u,0), (0,W))_{\mathbb{H}} \\
=& \sum_{ij\in E} \langle df(*e_{ij}) du(e_{ij}), W_{ij}\rangle \\
=& - \sum_{i\in V} \sum_{ij\in E:i} u_i \langle df(*e_{ij}), W_{ij}\rangle, \\
((0,Z), \mathrm{D}^*(0,W))_{\mathbb{H}} =& (\mathrm{D}(0,Z), (0,W))_{\mathbb{H}} \\
=& \sum_{ij\in E} \langle -df(e_{ij}) \times dZ(*e_{ij}), W_{ij}\rangle \\
=& \sum_{ij\in E} -\langle dZ(*e_{ij}), W_{ij} \times df(e_{ij})\rangle \\
=& \sum_{ijk\in F} -\langle Z_{ijk}, W_{ij} \times df(e_{ij}) + W_{jk} \times df(e_{jk}) + W_{ki} \times df(e_{ki})\rangle
\end{aligned}$$

By the linearity of D^*, we have

Lemma 3.22. *The adjoint* D^* *of the discrete Dirac operator* D *is the map* $\mathrm{D}^* : \mathbb{R}^{V+2E} \to (\mathbb{R}^{V+3F})^*$

$$\begin{pmatrix} u \\ W \end{pmatrix} \mapsto \begin{pmatrix} i \mapsto -\sum_{ij\in E} \langle df(*e_{ij}), W_{ij}\rangle \\ ij \mapsto \begin{array}{c} df(e_{ij}) \times W_{ij} + df(e_{jk}) \times W_{jk} + df(e_{ki}) \times W_{ki} \\ -\big(\alpha_i df(e_{jk}) + \alpha_j df(e_{ki}) + \alpha_k df(e_{ij})\big) \end{array} \end{pmatrix}.$$

In the following, given any vector U on edge e, write $U^\perp$ as the component of U orthogonal to $df(e)$.

Lemma 3.23. *The kernel of* D^* *contains constant functions in the sense of the proof and*

$$\dim(\operatorname{Ker} \mathrm{D}^*) \geq 4.$$

Proof. Let α be a constant real valued function on vertices, W be a constant $\mathbb{R}^3$-vector on edges. Then $(\alpha, W^\perp) \in \mathbb{R}^{V+2E}$ and

$$\begin{aligned}
\big(\operatorname{Re} \mathrm{D}^*(\alpha, W^\perp)\big)_i =& - \sum_{ij\in E} \langle df(*e_{ij}), W^\perp_{ij}\rangle \\
=& - \sum_{ij\in E} \langle df(*e_{ij}), W_{ij}\rangle \\
=& 0, \\
\big(\operatorname{Im} \mathrm{D}^*(\alpha, W^\perp)\big)_{ijk} =& df(e_{ij}) \times W^\perp_{ij} + df(e_{jk}) \times W^\perp_{jk} + df(e_{ki}) \times W^\perp_{ki}
\end{aligned}$$

$$\begin{aligned}&-\big(\alpha_i df(e_{jk})+\alpha_j df(e_{ki})+\alpha_k df(e_{ij})\big)\\ =&df(e_{ij})\times W_{ij}+df(e_{jk})\times W_{jk}+df(e_{ki})\times W_{ki}\\ &-\big(\alpha_i df(e_{jk})+\alpha_j df(e_{ki})+\alpha_k df(e_{ij})\big)\\ =&0.\end{aligned}$$

Suppose given a constant vector W as above with $W^{\perp}\equiv 0$, it implies W_e is parallel to $df(e)$ for every edge e. Since $\{df(e)|e\in E\}$ span at least two dimension subspace of $\mathbb{R}^3$, it implies $W\equiv 0$. Hence, the kernel of D^* contains constant functions in the above sense and its dimension is at least 4. ☐

Lemma 3.24. *Denote* $\operatorname{Ker}\mathrm{D}^a\subset(\mathbb{R}^{V+3F})^*$ *as the annihilator of* $\operatorname{Ker}\mathrm{D}$*, i.e. the set of linear functionals on* $\mathbb{R}^{V+3F}$ *which vanish on* $\operatorname{Ker}\mathrm{D}$*. Similarly, we have* $(\operatorname{Ker}\mathrm{D}^*)^a$*. Then*

$$\operatorname{Im}\mathrm{D}^*=(\operatorname{Ker}\mathrm{D})^a\quad,\quad\operatorname{Im}\mathrm{D}=(\operatorname{Ker}\mathrm{D}^*)^a,$$

and

$$\dim(\operatorname{Ker}\mathrm{D})=\dim(\operatorname{Ker}\mathrm{D}^*).$$

Proof. Notice that

$$((u,z),D^*(\alpha,W))_{\mathbb{H}}=(D(u,z),(\alpha,W))_{\mathbb{H}}.$$

We have

$$\operatorname{Im}\mathrm{D}^*\subseteq(\operatorname{Ker}\mathrm{D})^a\quad,\quad\operatorname{Im}\mathrm{D}\subseteq(\operatorname{Ker}\mathrm{D}^*)^a.$$

It implies

$$\begin{aligned}\dim(\operatorname{Im}\mathrm{D}^*)\le\dim((\operatorname{Ker}\mathrm{D})^a)&=V+3F-\dim(\operatorname{Ker}\mathrm{D})\\&=V+2E-\dim(\operatorname{Ker}\mathrm{D})\\&=\dim(\operatorname{Im}\mathrm{D})\end{aligned}$$

and

$$\begin{aligned}\dim(\operatorname{Im}\mathrm{D})\le\dim((\operatorname{Ker}\mathrm{D}^*)^a)&=V+2E-\dim(\operatorname{Ker}\mathrm{D}^*)\\&=V+3F-\dim(\operatorname{Ker}\mathrm{D}^*)\\&=\dim(\operatorname{Im}\mathrm{D}^*).\end{aligned}$$

Hence all the equalities hold and the claim is proved. ☐

Lemma 3.25. *Suppose* $\dim(\operatorname{Ker}\mathrm{D}^*)=4$*, then for all* $\dot\rho\in\mathbb{R}^V$ *and* $U\in\mathbb{R}^{2E}$*,*

$$\sum_{i\in V}\dot\rho=0\ \textit{and}\ \sum_{e\in E}U_e=0\Leftrightarrow(\dot\rho,U)\in\operatorname{Im}\mathrm{D}\,.$$

Proof. Suppose there exists $(u,Z)\in\mathbb{R}^{V+3F}$ such that $D(u,Z)=(\dot\rho,U)$. Since the image of $D(u,Z)$ is exact 2-form, its summation over the surface is zero. Hence

$$\sum_{i\in V}\dot\rho=0\text{ and }\sum_{e\in E}U_e=0.$$

For another direction, the assumption $\dim(\operatorname{Ker}\mathrm{D}^*)=4$ implies $\operatorname{Ker}\mathrm{D}^*$ consists of exactly constants function. Therefore,

$$\sum_{i\in V}\dot\rho=0\text{ and }\sum_{e\in E}U_e=0$$

implies $(\dot\rho,U)\in(\operatorname{Ker}\mathrm{D}^*)^a=\operatorname{Im}\mathrm{D}$. ☐

Given $(u, Z) \in \mathbb{R}^{V+3F}$ with $\mathrm{D}(u, Z) = (\dot{\rho}, 0)$, we define $W \in \mathbb{R}^{2E}$ as

$$\dot{\lambda}_{e_{jk}} = \frac{u_j + u_k}{2} + Z_\varphi + \frac{\cot\beta_i}{2}(u_k - u_j)N_\varphi =: \frac{u_j + u_k}{2} + \frac{W_{jk}}{2}.$$

Then,

$$W_e - W_{-e} = 2(\dot{\lambda}_e - \dot{\lambda}_{-e}) /\!/ df(e).$$

Hence, $W_e^\perp = W_{-e}^\perp$ and $W^\perp \in \mathbb{R}^{2E}$ is well defined.

Theorem 3.26. *Given $(u, Z) \in \mathbb{R}^{V+3F}$ with $\mathrm{D}(u, Z) = (\dot{\rho}, 0)$, Then,*

$$(u, W^\perp) \in \mathbb{R}^{V+2E}$$

and

$$\mathrm{D}^*(u, W^\perp) = (\dot{\rho}, 0).$$

The converse is also true. Given $(u, W) \in \mathbb{R}^{V+2E}$ with $\mathrm{D}^*(u, W) = (\dot{\rho}, 0)$, there exists unique $(u, Z) \in \mathbb{R}^{V+3F}$ such that for any edge $e \subset \phi \in F$,

$$\frac{W_e}{2} = (Z_\varphi + \frac{\cot\beta_i}{2}(\dot{u}_k - \dot{u}_j)N_\varphi)^\perp.$$

The element (u, Z) defined as above satisfies

$$\mathrm{D}(u, Z) = (\dot{\rho}, 0).$$

Hence, the solutions of $\operatorname{Im}\mathrm{D}(u, Z) = 0$ correspond one-to-one to the solutions of $\operatorname{Im}\mathrm{D}(u, W) = 0$.

Proof. Given $(u, Z) \in \mathbb{R}^{V+3F}$ with $(D(u, Z)) = (\dot{\rho}, 0)$, from formula (3.7) we define W on edges via

$$\dot{\lambda}_{e_{jk}} = \frac{u_j + u_k}{2} + Z_\varphi + \frac{\cot\beta_i}{2}(u_k - u_j)N_\varphi =: \frac{u_j + u_k}{2} + \frac{W_{jk}}{2}$$

The assumption $\operatorname{Im}(D(u, Z)) = 0$ implies

$$(u, W^\perp) \in \mathbb{R}^{V+2E}.$$

Recall that the construction of $\dot{\lambda}$ satisfies

$$\begin{aligned}
0 =& \operatorname{Im}(df(e_{ij})\dot{\lambda}_{e_{ij}} + df(e_{jk})\dot{\lambda}_{e_{jk}} + df(e_{ki})\dot{\lambda}_{e_{ki}}) \\
=& df(e_{ij}) \times \frac{W_{ij}}{2} + df(e_{jk}) \times \frac{W_{jk}}{2} + df(e_{ki}) \times \frac{W_{ki}}{2} \\
&+ \frac{u_i + u_j}{2} df(e_{ij}) + \frac{u_j + u_k}{2} df(e_{jk}) + \frac{u_k + u_i}{2} df(e_{ki}) \\
=& df(e_{ij}) \times \frac{W_{ij}}{2} + df(e_{jk}) \times \frac{W_{jk}}{2} + df(e_{ki}) \times \frac{W_{ki}}{2} \\
&- \frac{u_i}{2} df(e_{jk}) - \frac{u_j}{2} df(e_{ki}) - \frac{u_k}{2} df(e_{ij}).
\end{aligned}$$

Hence,

$$\operatorname{Im}\mathrm{D}^*((u, W^\perp)) \equiv 0.$$

We now want to compute $\operatorname{Re}\mathrm{D}^*((u, W^\perp))$. Notice that

$$W_{ji} - W_{ij} = 2(\dot{\lambda}_{ji} - \dot{\lambda}_{ij}) = -\dot{\alpha_{ij}} T_{e_{ij}}.$$

Also, in a triangle $ijk \in F$,

$$\frac{W_{ij}}{2} - \frac{W_{ki}}{2} = \big(\frac{\cot\beta_k}{2}(u_j - u_i) - \frac{\cot\beta_j}{2}(u_k - u_j)\big)N_{ijk}.$$

So,

$$\langle W_{ij} - W_{ki}, df(\phi_{i,ijk}) \rangle = 0$$

where $df(\phi_{i,ijk})$ denote the vector from vertex i to the circumcenter of triangle (ijk) under the image of f. Summing it around a vertex i,

$$\begin{aligned} 0 &= \sum_{ijk\in F:i} \langle W_{ij} - W_{ki}, df(\phi_{i,ijk})\rangle \\ &= \sum_{ij\in E:i} \langle W_{ij}, df(\phi_{i,ijk})\rangle - \langle W_{ji}, df(\phi_{i,ij\tilde{k}})\rangle \\ &= \sum_{ij\in E:i} \langle W_{ij}, df(\phi_{i,ijk}) - df(\phi_{i,ij\tilde{k}})\rangle + \langle W_{ij} - W_{ji}, df(\phi_{i,ij\tilde{k}})\rangle \\ &= \sum_{ij\in E:i} \langle W_{ij}, df(*e_{ij})\rangle + \langle \dot{\alpha}_{ij} T_{e_{ij}}, df(\phi_{i,ij\tilde{k}})\rangle. \end{aligned}$$

Hence,

$$\begin{aligned} (\operatorname{Re}\mathrm{D}^*((u, W^\perp)))_i &= -\sum_{ij\in E:i} \langle W_{ij}, df(*e_{ij})\rangle \\ &= \sum_{ij\in E:i} \langle \dot{\alpha}_{ij} T_{e_{ij}}, \frac{df(e_{ij})}{2} + \frac{\cot\beta_{\tilde{k}}}{2} df(e_{ij}) \times N_{ij\tilde{k}}\rangle \\ &= \sum_{ij\in E:i} \frac{\dot{\alpha}}{2} |df(e_{ij})| \\ &= \dot{\rho}_i. \end{aligned}$$

For the converse, since $\operatorname{Im}\mathrm{D}^*(u, W^\perp) = 0$,the existence and uniqueness of (u, Z) with $\operatorname{Im}\mathrm{D}(u, Z) = 0$ is ensured by Theorem (3.10). And from the first part of the theorem, we have

$$\operatorname{Re}\mathrm{D}(u, Z) = \operatorname{Re}\mathrm{D}^*(u, W^\perp).$$

□

Remark 3.27. *The above result should be expected since the quaternion Dirac operator in the smooth theory is self adjoint.*

Remark 3.28. *With the theorem, one could calculate under an infinitesimal conformal deformation* $\dot{\rho}_i = \sum_{ij\in E:i} \frac{\dot{\alpha}_{ij}}{2} |df(e)|$ *directly by* $\dot{\lambda}$ *defined on edges, without representing it as* $(u, Z) \in \mathbb{R}^{V+3F}$.

4. Infinitesimal Conformal Deformations of High Genus Surfaces

In Remark 3.7, we have heuristic argument that the dimension of immersed triangulated surfaces in a given conformal class is $(3V - 7) - (E - V) = V - 1 - 6g$. The following shows its orthogonal complement in the space of $\mathbb{R}$-valued functions on vertices.

In the following, we implicitly use an identification between discrete 1-forms and dual 1-forms in order to apply Hodge decomposition theorem (1.19). The result turns out to be independent of the identification used.

Theorem 3.29. *Suppose* $f : M \to \mathbb{R}^3$ *is an immersion of a triangulated surface* M *of genus* g *with* $\dim(\operatorname{Ker}\mathrm{D}) = 4$ *and* $\omega_1, \ldots, \omega_{2g}$ *form a basis of harmonic 1-forms. Let* $\tilde{e}_1 := (1, 0, 0)$,$\tilde{e}_2 = (0, 1, 0)$,$\tilde{e}_3 = (0, 0, 1)$.

Then for all $k = 1, 2, \ldots, 2g$ *and* $l = 1, 2, 3$, *there exists* $(u_{kl}, Z_{kl}) \in \mathbb{R}^{V+3F}$ *such that*

$$\mathrm{D}(u_{kl}, Z_{kl}) = \begin{pmatrix} i \mapsto \sum_{ij\in E:i} \langle \tilde{e}_l, \omega_k(*e_{ij}) df(e_{ij})\rangle \\ ij \mapsto \omega_k(*e_{ij}) \tilde{e}_l \times df(e_{ij}) \end{pmatrix}.$$

And given $(u,Z)\in\mathbb{R}^{V+3F}$ *with* $\mathrm{D}(u,Z)=(\dot\rho,0)$*, we have*

$$2\operatorname{Im}(df\dot\lambda)\text{ is exact}\Leftrightarrow\sum_{i\in V}\dot\rho_i u_{kl,i}=0\quad\forall k=1,2,\dots,2g\text{ and }l=1,2,3,$$

where $\dot\lambda$ *is given by the formula* (3.7)*.*

Hence, combined with Lemma 3.25, the vector space

$$\{\dot\rho\in\mathbb{R}^V|\sum\dot\rho=0,\sum\dot\rho u_{kl}=0\quad\forall k=1,2,\dots,2g\text{ and }l=1,2,3\}$$

is the space of all nontrivial infinitesimal conformal deformations of f*, i.e. up to Euclidean transformations.*

Proof. Since ω_k is a harmonic 1-form and hence co-closed,

$$\begin{aligned}\sum_{i\in V}\sum_{ij\in E:i}\langle\tilde e_l,\omega_k(*e_{ij})df(e_{ij})\rangle&=2\sum_{ij\in E}\langle\tilde e_l,\omega_k(*e_{ij})df(e_{ij})\rangle\\&=2\sum_{j\in V}\langle\sum_{ij\in E:j}\omega_k(*e_{ij})\tilde e_l,f_j\rangle\\&=0.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{ij\in E}\omega_k(*e_{ij})\tilde e_l\times df(e_{ij})&=\sum_{j\in V}\big(\sum_{ij\in E:j}\omega_k(*e_{ij})\big)\tilde e_l\times f_j\big)\\&=0.\end{aligned}$$

By Lemma (3.25), the sums being zero imply the existence of (u_{kl},Z_{kl}) as claimed.

Given $(u,Z)\in\mathbb{R}^{V+3F}$ with $\mathrm{D}(u,Z)=(\dot\rho,0)$, define $(u,W^\perp)\in\mathbb{R}^{V+2E}$ as previously via

$$\dot\lambda_{e_{jk}}=\frac{u_j+u_k}{2}+Z_\varphi+\frac{\cot\beta_i}{2}(u_k-u_j)N_\varphi=:\frac{u_j+u_k}{2}+\frac{W_{jk}}{2}.$$

The condition $\operatorname{Im}(D(u,Z))=0$ implies $2\operatorname{Im}(df\dot\lambda)$ is a closed discrete 1-form. By Hodge decomposition theorem (1.19), we have

$$\begin{aligned}2\operatorname{Im}(df\dot\lambda)\text{ is exact}&\Leftrightarrow\sum_{e\in E}2\operatorname{Im}(df\dot\lambda)\omega_k(*e)=0\quad\forall k=1,2,3,\dots,2g\\&\Leftrightarrow\sum_{e\in E}\langle\tilde e_l,2\operatorname{Im}(df\dot\lambda)\omega_k(*e)\rangle=0\quad\forall k=1,2,3,\dots,2g,l=1,2,3.\end{aligned}$$

Then, for all $k=1,2,3,\dots,2g,l=1,2,3$

$$\begin{aligned}0&=\sum_{e\in E}\langle\tilde e_l,2\operatorname{Im}(df\dot\lambda)\omega_k(*e)\rangle\\&=\sum_{ij\in E}\langle\tilde e_l,\omega_k(*e_{ij})\big(df(e_{ij})\times W_{e_{ij}}+(u_i+u_j)df(e_{ij})\big)\rangle\\&=\sum_{i\in V}\sum_{ij\in E:i}\langle\tilde e_l,\omega_k(*e_{ij})df(e_{ij})\rangle u_i+\sum_{ij\in E}\langle\omega_k(*e_{ij})\tilde e_l\times df(e_{ij}),W_{ij}\rangle\\&=(\mathrm{D}(u_{kl},Z_{kl}),(u,W))_{\mathbb{H}}\\&=((u_{kl},Z_{kl}),\mathrm{D}^*(u,W))_{\mathbb{H}}\\&=((u_{kl},Z_{kl}),(\dot\rho,0))_{\mathbb{H}}\\&=\sum_{i\in V}\dot\rho_i u_{kl,i}.\end{aligned}$$

□

These 6g equations and the constant function $\mathbf{1}$ are not always linearly independent. As in the smooth theory (Theorem 2.21), their linearly dependence is equivalent to the existence of some special closed 1-form. The proof for the discrete case is the same as in the smooth case. The only difference is that the representation of quaternion functions $(u, Z) \in \mathbb{R}^{V+3F}$ here is only $\mathbb{R}$-linear instead of $\mathbb{H}$-linear. It makes the computation complicated.

Theorem 3.30. *Suppose M is a closed triangulated surface with genus g and f is an immersion with $\dim(\operatorname{Ker} \mathrm{D}) = 4$. Then the 6g equations u_{kl} in Theorem 3.29 and the constant function $\mathbf{1}$ defined on vertices are $\mathbb{R}$-linearly dependent if and only if there exist a $\mathbb{R}^3$-valued co-closed discrete 1-form τ such that*

$$df(e_{ij}) \times \tau(*e_{ij}) = 0 \quad \forall e_{ij} \in E,$$
$$\sum_{ij \in E:i} \langle df(e_{ij}), \tau(*e_{ij}) \rangle = 0 \quad \forall i \in V.$$

Proof. Suppose there exists a $\mathbb{R}^3$-valued co-closed discrete 1-form τ such that

$$df(e_{ij}) \times \tau(*e_{ij}) = 0 \quad \forall e_{ij} \in E,$$
$$\sum_{ij \in E:i} \langle df(e_{ij}), \tau(*e_{ij}) \rangle = 0 \quad \forall i \in V.$$

The co-closed 1-form can be integrated over dual graph of the triangulation. Denote $a_1, b_1, a_2, b_2, \dots, a_g, b_g$ be closed cycles on the dual graph representing a canonical basis of its homology and let $\omega_1, \omega_2, \dots, \omega_{2g}$ be a basis of discrete harmonic 1-forms such that $\sum_{a_j} \omega_i = \sum_{b_j} \omega_{g+i} = \delta_{ij}$ and $\sum_{a_j} \omega_{g+i} = \sum_{b_j} \omega_i = 0$. Define constant $\mathbb{R}^3$ vectors $A_i := \sum_{a_i} \tau$ and $B_i := \sum_{b_i} \tau$. Then,

$$\tau - \sum_{k=1}^{g} A_k \omega_k - B_k \omega_{g+k}$$

is a $\mathbb{R}^3$-valued co-exact dual 1-form. And hence there exists a dual 0-form μ such that $du = \tau - \sum_{k=1}^{2g} A_k \omega_k - B_k \omega_k$. Then we have on any edge e_{ij}

$$df(e_{ij}) \times du(*e_{ij}) = -\sum_{k=1}^{g} df(e_{ij}) \times A_k \omega_k(*e_{ij}) + df(e_{ij}) \times B_k \omega_{g+k}(*e_{ij})$$

and on any vertex i

$$\sum_{ij \in E:i} \langle df(e_{ij}), du(*e_{ij}) \rangle = \sum_{ij \in E:i} \sum_{k=1}^{g} \langle df(e_{ij}), A_k \rangle \omega_k(*e_{ij}) + \langle df(e_{ij}), B_k \rangle \omega_{g+k}(*e_{ij}).$$

Hence, by writing $A_k = (A_k^1, A_k^2, A_k^3)$ and $B_k = (B_k^1, B_k^2, B_k^3)$, we have

$$\mathrm{D}\left((0, u) - \sum_{k=1}^{g} \sum_{l=1}^{3} A_k^l (u_{kl}, Z_{kl}) - \sum_{k=1}^{g} \sum_{l=1}^{3} B_k^l (u_{(g+k)l}, Z_{(g+k)l})\right) = 0.$$

Since $\dim \operatorname{Ker} \mathrm{D} = 4$, there exists $(c_0, c) \in \mathbb{R}^{1+3}$ such that

$$(0, u) - \sum_{k=1}^{g} \sum_{l=1}^{3} A_k^l (u_{kl}, Z_{kl}) - \sum_{k=1}^{g} \sum_{l=1}^{3} B_k^l (u_{(g+k)l}, Z_{(g+k)l}) \equiv (c_0, c).$$

It implies

$$\sum_{k=1}^{g} \sum_{l=1}^{3} A_k^l u_{kl} + \sum_{k=1}^{g} \sum_{l=1}^{3} B_k^l u_{(g+k)l} \equiv -c_0.$$

Hence, the $6g + 1$ equations are $\mathbb{R}$-linearly dependent.

We now prove another direction. Suppose there exists constants $A^l_k, B^l_k, c \in \mathbb{R}$ such that

$$\sum_{k=1}^{g}\sum_{l=1}^{3} A^l_k u_{kl} + \sum_{k=1}^{g}\sum_{l=1}^{3} B^l_k u_{(g+k)l} \equiv c.$$

Without loss of generality, we assume $c = 0$. Then we define

$$(0, Z) := \sum_{k=1}^{g}\sum_{l=1}^{3} A^l_k (u_{kl}, Z_{kl}) + \sum_{k=1}^{g}\sum_{l=1}^{3} B^l_k (u_{(g+k)l}, Z_{(g+k)l}) \in \mathbb{R}^{V+3F}$$

and a $\mathbb{R}^3$-valued dual 1-form

$$\tau = dZ - \sum_{k=1}^{g} A_k \omega_k - B_k \omega_{g+k}.$$

Since τ is a linearly combination of co-closed dual 1-forms, τ is also co-closed. In addition, on any edge e_{ij} and vertex i,

$$\begin{aligned} df(e_{ij}) \times \tau(*e_{ij}) =& df(e_{ij}) \times dZ(*e_{ij}) \\ & - \sum_{k=1}^{g} df(e_{ij}) \times A_k \omega_k(*e_{ij}) + df(e_{ij}) \times B_k \omega_{g+k}(*e_{ij}) \\ =& -\operatorname{Im}(\mathrm{D}(0,Z)) + \operatorname{Im}\Big(\mathrm{D}\,\big(\sum_{k=1}^{g}\sum_{l=1}^{3} A^l_k (u_{kl}, Z_{kl}) \\ & + \sum_{k=1}^{g}\sum_{l=1}^{3} B^l_k (u_{(g+k)l}, Z_{(g+k)l})\big)\Big) \\ =& 0 \end{aligned}$$

and

$$\begin{aligned} \sum_{ij\in E:i} \langle df(e_{ij}), \tau(*e_{ij})\rangle =& \sum_{ij\in E:i} \big(\langle df(e_{ij}), dZ(*e_{ij})\rangle \\ & - \sum_{k=1}^{g} \langle df(e_{ij}), A_k\rangle \omega_k(*e_{ij}) + \langle df(e_{ij}), B_k\rangle \omega_{g+k}(*e_{ij})\big) \\ =& \operatorname{Re}(\mathrm{D}(0,Z)) - \operatorname{Re}\Big(\mathrm{D}\,\big(\sum_{k=1}^{g}\sum_{l=1}^{3} A^l_k (u_{kl}, Z_{kl}) \\ & + \sum_{k=1}^{g}\sum_{l=1}^{3} B^l_k (u_{(g+k)l}, Z_{(g+k)l})\big)\Big) \\ =& 0. \end{aligned}$$

□

Generally, similar to the smooth case, we have conditions on $\dot{\lambda}$ to ensure the closedness and exactness of the discrete 1-form $2\operatorname{Im}(df\dot{\lambda})$. We start with the closedness condition. It is a special case of vector-valued Schläfli formula in Souam and Schlenker (2008) by considering infinitesimal deformations of closed polygons.

Corollary 3.31. *Suppose $(u, Z) \in \mathbb{R}^{V+3F}$ satisfies $\operatorname{Im}\mathrm{D}(u, Z) = 0$. Then,*

$$0 = \sum_{ijk\in F:i} \dot{\beta}_{i,ijk} N_{ijk} - \dot{\alpha}_{e_{ij}} T_{e_{ij}}$$

where $\beta_{i,ijk}$ is the face angle of triangle (ijk) at vertex i and $\alpha_{e_{ij}}$ is the dihedral angle at the edge e_{ij}.

PROOF. For any vertex i, summing $Z_{\tilde{\varphi}} - Z_{\varphi}$ over all the edges starting at it,

$$0 = \sum_{ij\in E:i} (u_j - u_i)(\frac{\cot\beta_{k,ijk}}{2} N_{ijk} + \frac{\cot\beta_{\tilde{k},i\tilde{k}j}}{2} N_{i\tilde{k}j}) - \dot{\alpha}_{e_{ij}} T_{e_{ij}}$$

$$= \sum_{ijk\in F:i} ((u_j - u_i)\frac{\cot\beta_{k,ijk}}{2} + (u_k - u_i)\frac{\cot\beta_{j,ijk}}{2}) N_{ijk} - \dot{\alpha}_{e_{ij}} T_{e_{ij}}.$$

From previous calculation,

$$\sigma_{e_{ij}} = \frac{u_i + u_j}{2}$$

and

$$\dot{\beta}_i = \omega_{ki} - \omega_{ij}$$
$$= -\cot\beta_j(\sigma_{ij} - \sigma_{jk}) + \cot\beta_k(\sigma_{jk} - \sigma_{ki})$$
$$= (u_k - u_i)\frac{\cot\beta_j}{2} + (u_j - u_i)\frac{\cot\beta_k}{2}.$$

Hence,

$$\sum_{ijk\in E:i} \dot{\beta}_{i,ijk} N_{ijk} - \dot{\alpha}_{e_{ij}} T_{e_{ij}} = 0.$$

□

We now investigate the exactness condition. Suppose we have an oriented triangulated surface. A closed discrete curve on the dual surface M^* is called a triangle strip.

Corollary 3.32. *Suppose we have an oriented triangulated surface $f : M \to \mathbb{R}^3$ with arbitrary genus. Given $(u, Z) \in \mathbb{R}^{V+3F}$ with $\operatorname{Im} \mathrm{D}(u, Z) = 0$, we have $\dot{\lambda}$ on oriented edges which describes changes of edge vector. Then, such a variation of edge vectors gives rise to an infinitesimal deformation of the surface, i.e.*

$$2\operatorname{Im}(df\dot{\lambda}) \text{ is exact}$$

if and only if for any triangle strip (See Figure 3.1)

$$\sum_{i=1}^{2n} \dot{\ell}_{i,i+1} T_{i,i+1} = \sum_i \gamma_i \times \Big(\sum_{\substack{ijk\in F:(ijk) \text{ is} \\ \text{between } T_{i-1,i} \text{ and} \\ T_{i,i+1} \text{ in the strip}}} \dot{\beta}_{ijk,i} N_{ijk} - \sum_{\substack{ij\in E:(ij) \text{ is} \\ \text{between left face of} \\ T_{i-1,i} \text{ and } T_{i,i+1} \\ \text{in the strip}}} \dot{\alpha}_{ij} T_{ij} \Big).$$

PROOF. Consider triangle strip S on M with $\partial S = C_1 \cup C_2$ where C_1, C_2 are closed discrete curves. Consider a zic-zac closed curve γ with vertices $\gamma_1, \gamma_2, \ldots, \gamma_{2n}$ as in Figure 3.1,

Notice that

$$\gamma_{i+1} - \gamma_i = \ell_{i,i+1} T_{i,i+1}$$
$$\dot{\big(\gamma_{i+1} - \gamma_i\big)} = \dot{\ell}_{i,i+1} T_{i,i+1} + \ell_{i,i+1} T_{i,i+1} \times \operatorname{Im}(\lambda_{T_{i,i+1}})$$
$$= \dot{l}_{i,i+1} T_{i,i+1} + (\gamma_{i+1} - \gamma_i) \times \operatorname{Im}(\lambda_{T_{i,i+1}}).$$

The curve C_1 remaining closed is equivalent to

$$0 = \sum_{i=1}^{n} \dot{\big(\gamma_{2i+1} - \gamma_{2i-1}\big)}$$
$$= \sum_{i=1}^{n} \dot{\big(\gamma_{2i+1} - \gamma_{2i} + \gamma_{2i} - \gamma_{2i-1}\big)}$$

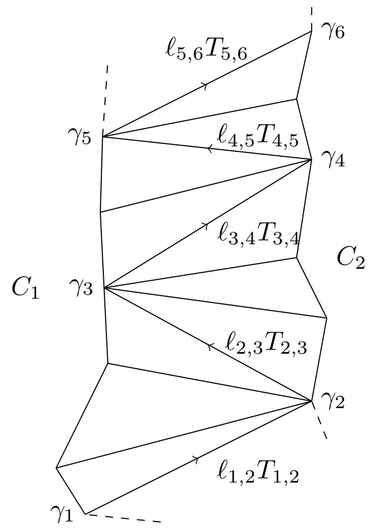

FIGURE 3.1. A triangle strip

$$
\begin{aligned}
&= \sum_{i=1}^{2n} \overset{\cdot}{\left(\gamma_{i+1} - \gamma_i\right)} \\
&= \sum_{i=1}^{2n} \dot{\ell}_{i,i+1} T_{i,i+1} + \sum_{i} \gamma_i \times \left(\operatorname{Im}(\lambda_{T_{i-1,i}}) - \operatorname{Im}(\lambda_{T_{i,i+1}})\right) \\
&= \sum_{i=1}^{2n} \dot{\ell}_{i,i+1} T_{i,i+1} - \sum_{i} \gamma_i \times \Big(\sum_{\substack{ijk \in F : (ijk) \text{ is} \\ \text{between } T_{i-1,i} \text{ and} \\ T_{i,i+1} \text{ in the strip}}} \dot{\beta}_{ijk,i} N_{ijk} - \sum_{\substack{ij \in E : (ij) \text{ is} \\ \text{between left face of} \\ T_{i-1,i} \text{ and } T_{i,i+1} \\ \text{in the strip}}} \dot{\alpha}_{ij} T_{ij} \Big)
\end{aligned}
$$

□

5. Vector Analysis of the Discrete Dirac Operator

In the smooth theory, $\operatorname{Re}(\mathrm{D}^2)$ on real valued functions is the Laplace operator as shown in Corollary 2.27. The same form is considered for the discrete Dirac operator. It coincides up to a positive constant with the cotangent Laplace operator, which is a famous discretized Laplace operator on triangulated surfaces introduced by Pinkall and Polthier (1993).

Theorem 3.33. *For any real-valued function defined on vertices $\alpha : V \to \mathbb{R}$, the real part of $\mathrm{D}\,\frac{1}{A}\,\mathrm{D}^*(\alpha, 0)$ on vertices is given by*

$$
(\mathrm{D}\,\frac{1}{A}\,\mathrm{D}^*(\alpha, 0))_i = -2 \sum_{ij \in E : i} (\cot \beta_k + \cot \beta_{\tilde{k}})(\alpha_j - \alpha_i)
$$

and the imaginary part on edges is given by

$$
(\mathrm{D}\,\frac{1}{A}\,\mathrm{D}^*(\alpha, 0))_{ij} = 2(\alpha_j - \alpha_i)(N_{ijk} - N_{i\tilde{k}j}),
$$

where $A : F \to \mathbb{R}^+$ is the area of the corresponding triangle under the immersion and $(ijk), (ij\tilde{k}) \in F$ are faces sharing an edge (ij).

PROOF. We first consider the real part on vertices.

$$
\mathrm{D}^*(\alpha, 0)_{ijk} = -\left(\alpha_i df(e_{jk}) + \alpha_j df(e_{ki}) + \alpha_k df(e_{ij})\right)
$$

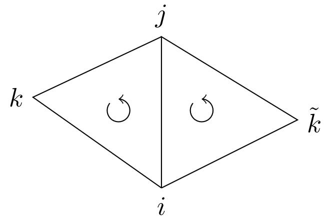

FIGURE 3.2. Two neighboring triangles containing edge e_{ij}

$$
\begin{aligned}
(\mathrm{D}\,\frac{1}{A}\,\mathrm{D}^*(\alpha,0))_i &= \sum_{ij\in E:i}\langle -df(e_{ij}),\,\frac{1}{A_{ji\tilde{k}}}\,\mathrm{D}^*(\alpha,0)_{ji\tilde{k}} - \frac{1}{A_{jik}}\,\mathrm{D}^*(\alpha,0)_{ijk}\rangle \\
&= \sum_{ij\in E:i}\langle df(e_{ij}),\,\frac{\alpha_{\tilde{k}}df(e_{ji}) + \alpha_j df(e_{i\tilde{k}}) + \alpha_i df(e_{\tilde{k}j})}{A_{ji\tilde{k}}} \\
&\quad - \frac{\alpha_i df(e_{jk}) + \alpha_j df(e_{ki}) + \alpha_k df(e_{ij})}{A_{ijk}}\rangle,
\end{aligned}
$$

where A_{ijk} is the area of triangle (ijk). Notice that

$$
\begin{aligned}
\text{coefficient of } \alpha_j =& \frac{\langle df(e_{ij}), df(e_{i\tilde{k}})\rangle}{A_{ij\tilde{k}}} + \frac{\langle df(e_{ij}), df(e_{ik})\rangle}{A_{ijk}} \\
& - \frac{\langle df(e_{i\tilde{k}}), df(e_{i\tilde{k}})\rangle}{A_{ij\tilde{k}}} - \frac{\langle df(e_{ik}), df(e_{ik})\rangle}{A_{ijk}} \\
=& \frac{\langle df(e_{ij}) - df(e_{i\tilde{k}}), df(e_{i\tilde{k}})\rangle}{A_{ij\tilde{k}}} - \frac{\langle df(e_{ik}), df(e_{ij}) - df(e_{ik})\rangle}{A_{ijk}} \\
=& \frac{\langle df(e_{\tilde{k}j}), df(e_{i\tilde{k}})\rangle}{A_{ij\tilde{k}}} + \frac{\langle df(e_{ik}), df(e_{kj})\rangle}{A_{ijk}} \\
=& -2(\cot\beta_{\tilde{k}} + \cot\beta_k) \\
\text{and coefficient of } \alpha_i =& \sum_{ijk\in F:i} \frac{\langle df(e_{ij}), -df(e_{jk})\rangle + \langle df(e_{ik}), df(e_{jk})\rangle}{A_{ijk}} \\
=& \sum_{ijk\in F:i} 2(\cot\beta_j + \cot\beta_k) \\
=& \sum_{ij\in E:i} 2(\cot\beta_k + \cot\beta_{\tilde{k}}).
\end{aligned}
$$

Hence, the real part has formula

$$
(\mathrm{D}\,\frac{1}{A}\,\mathrm{D}^*(\alpha,0))_i = -2\sum_{ij\in E:i}(\cot\beta_k + \cot\beta_{\tilde{k}})(\alpha_j - \alpha_i).
$$

Now we consider the imaginary part on edges.

$$
\begin{aligned}
(\mathrm{D}\,\frac{1}{A}\,\mathrm{D}^*(\alpha,0))_{ij} =& df(e_{ij}) \times \big(- \frac{\alpha_i df(e_{jk}) + \alpha_j df(e_{ki}) + \alpha_k df(e_{ij})}{A_{ijk}} \\
& + \frac{\alpha_{\tilde{k}} df(e_{ji}) + \alpha_j df(e_{i\tilde{k}}) + \alpha_i df(e_{\tilde{k}j})}{A_{ji\tilde{k}}}\big) \\
=& - df(e_{ij}) \times \big(\frac{\alpha_i df(e_{jk}) + \alpha_j df(e_{ki}) +}{A_{ijk}} + \frac{\alpha_j df(e_{i\tilde{k}}) + \alpha_i df(e_{\tilde{k}j})}{A_{ji\tilde{k}}}\big)
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{-\alpha_i|df(e_{ij})||df(e_{jk})|\sin\beta_j+\alpha_j|df(e_{ki})||df(e_{ij})|\sin\beta_i}{A_{ijk}}N_{ijk}\\
&+\frac{-\alpha_j|df(e_{ij})||df(e_{i\tilde{k}})|\sin\tilde{\beta}_i+\alpha_i|df(e_{ij})||df(e_{j\tilde{k}})|\sin\tilde{\beta}_j}{A_{ji\tilde{k}}}N_{i\tilde{k}j}\\
=&2(\alpha_j-\alpha_i)(N_{ijk}-N_{i\tilde{k}j}).
\end{aligned}
$$

□

Remark 3.34. *The formula above should be compared with that in the smooth case. That is, given an immersion f, for any real valued function $g : M \to \mathbb{R}$, we have* $\mathrm{D}^2\, g = \Delta g - df(\mathrm{A}\, J \operatorname{grad} g)$*, where* A *is the shape operator of f.*

Remark 3.35. *One may wonder if there is a formula of* $\mathrm{D}\,\frac{1}{A}\,\mathrm{D}^*$ *on the other components. Instead, it is not clear since there is no natural choice for the tangential and normal component of a vector living on an edge. On the other hand, the formula of* $\mathrm{Re}(\mathrm{D}^*\,\frac{1}{A}\,\mathrm{D}(u,0))$ *is also not clear, since there is no natural choice for the area A of an edge.*

CHAPTER 4

Examples

This chapter considers various examples. Section 1-2 compares infinitesimal conformal deformations of the unit sphere $\mathbb{S}^2$ in the smooth case and the discrete case. It shows that convex triangulated surfaces with vertices on the unit sphere behave similar to the smooth unit sphere under infinitesimal conformal deformations. It serves a positive evidence that the definition of conformal equivalence of triangulated surfaces makes sense. It is a result from the blog "Discrete Spin Geometry".

Section 3 considers double inversions in the smooth and the discrete case. The explicit formula for $\dot{\rho}$ in two cases have very nice correspondence. It justifies the definition of $\dot{\rho}$ in Theorem 3.10.

Section 4 is concerned with the kernel of the discrete Dirac operator. In the smooth theory, those conformal immersions of a surface with the dimension of kernel larger than 4 are singular points of the shape space under the parametrization by mean curvature half-density. These immersions include constant mean curvature immersions and constrained Willmore surfaces. Considering the kernel of the discrete Dirac operator would give hints to their discrete analogue. Both examples of surfaces with dim(Ker(D)) = 4 and dim(Ker(D)) > 4 are given.

1. Infinitesimal Conformal deformations of Smooth Spheres

In order to compare with conformal deformations of a discrete round sphere, we need to know the case for a smooth round sphere. Let $f : S^2 \to \mathbb{R}^3$ be the standard smooth unit sphere.

Theorem 4.1. *Up to infinitesimal rotations and translations, the infinitesimal conformal deformations of the unit sphere are infinitesimal normal deformations.*

Proof. We first consider an infinitesimal normal variation in the form

$$\dot{f} = vN$$

where v is some arbitrary real-valued function on S^2. For the unit sphere, the shape operator $A : TS^2 \to TS^2$ is an identity operator. Then for any vector $X, Y \in TS^2$,

$$\begin{aligned}\dot{\langle df(X), df(Y)\rangle} =& \langle d\dot{f}(X), df(Y)\rangle + \langle df(X), d\dot{f}(Y)\rangle \\ =& \langle vdN(X), df(Y)\rangle + \langle dv(X)N, df(Y)\rangle \\ &+ \langle df(X), vdN(Y)\rangle + \langle df(X), dv(Y)N\rangle \\ =& \langle vdf(\mathrm{A}\,X), df(Y)\rangle + \langle df(X), vdf(\mathrm{A}\,Y)\rangle \\ =& 2v\langle df(X), df(Y)\rangle.\end{aligned}$$

Hence, all infinitesimal normal deformations are conformal. And there exist $\dot{\lambda}$ such that

$$dvN + vdf = d\dot{f} = (\dot{\overline{\lambda}df\lambda}) = 2(gdf - df(Y) \times df - hN \times df),$$

where $\dot{\lambda} = g + df(Y) + hN$ is decomposed into scalar, tangential and normal components. Since for any tangent vector $X \in TS^2$

$$df(Y) \times df(X) = \langle Y, -JX\rangle N = \langle JY, X\rangle N,$$

comparing the coefficients yields

$$\begin{aligned} g &= \frac{v}{2}, \\ Y &= \frac{1}{2} J \operatorname{grad} v, \\ h &= 0. \end{aligned}$$

It is observed that given any function g, there exists an infinitesimal normal variation with prescribed real part of $\dot{\lambda}$. It implies the infinitesimal deformations with $g \equiv 0$ a complementary subspace to normal variations. Suppose an infinitesimal conformal deformation is given by $\dot{\lambda} = g + df(Y) + hN$. Then there exists $\dot{\rho} : S^2 \to \mathbb{R}$ such that

$$\mathrm{D}\,\dot{\lambda} = \dot{\rho}.$$

From Theorem 2.26, it is equivalent to

$$\begin{aligned} -\operatorname{curl} Y &= \dot{\rho}, \\ Y &= J \operatorname{grad} g + \operatorname{grad} h, \\ 2h &= -\operatorname{div} Y. \end{aligned}$$

The condition $g \equiv 0$ implies $Y = \operatorname{grad} h$ and hence

$$\Delta h = -2h.$$

So, h is a spherical harmonic function and is of the form

$$h = \langle a, f\rangle$$

for some constant vector $a \in R^3$. The equation $Y = \operatorname{grad} h$ implies Y is tangential projection of the constant vector a. Hence,

$$\dot{\lambda} = a,$$

which gives a trivial infinitesimal conformal deformation. □

Remark 4.2. *It is known that Möbius transformations are conformal transformations of the ambient space. They are generated by translations, rotations, scalings and the inversion under the unit sphere. The infinitesimal transformations of the unit sphere generated by inversions should be included in the above lemma. Infinitesimal conformal deformations via double inversions (Section 3) are in the forms*

$$\dot{f} = -faf = -a||f||^2 + 2f\langle a, f\rangle$$

where a is a constant vector. For the unit sphere, we have $||f|| = 1$. And so we get

$$\dot{f} = -faf = -a + 2f\langle a, f\rangle.$$

Hence, the infinitesimal conformal deformations via double inversions are composition of a translation and a normal deformation.

2. Conformal Deformations of Discrete Round Sphere

Let $f : M \to \mathbb{R}^3$ be a triangulated surface circumscribed on the unit sphere $\mathbb{S}^2$. Consider a "normal deformation"' of the form

$$\dot{f}_i = \nu_i f_i.$$

Notice that

$$df(e_{ij}) = f_j - f_i.$$

Differentiating both sides yields

$$\begin{aligned} d\dot{f}(e_{ij}) &= \dot{f}_j - \dot{f}_i \\ &= \nu_j f_j - \nu_i f_i \\ &= \frac{\nu_i + \nu_j}{2} df(e_{ij}) + (\nu_j - \nu_i)\frac{f_i + f_j}{2}. \end{aligned}$$

Since $|f_i| = |f_j| = 1$, we have

$$\frac{|d\dot{f}(e_{ij})|}{|df(e_{ij})|} = \frac{\langle |d\dot{f}(e_{ij})|, |df(e_{ij})|\rangle}{\langle |df(e_{ij})|, |df(e_{ij})|\rangle} = \frac{\nu_i + \nu_j}{2}.$$

Hence, the normal variation is always conformal. Denote f_ϕ as the circumcenter of the face $\phi = (ijk)$. Then,

$$f_\phi = \cos\rho_\phi N_\phi$$

where

$$\cos\rho_\phi = \langle f_i, N_\phi\rangle = \langle f_j, N_\phi\rangle = \langle f_k, N_\phi\rangle.$$

By elementary geometry,

$$\frac{f_i + f_j}{2} = f_\phi + \frac{\cot\beta_k}{2} df(e_{ij}) \times N_\phi.$$

Then,

$$d\dot{f}(e_{ij}) = \frac{\nu_i + \nu_j}{2} df(e_{ij}) + (\nu_j - \nu_i)(\cos\rho_\phi N_\phi + \frac{\cot\beta_k}{2} df(e_{ij}) \times N_\phi).$$

Note that

$$\begin{aligned} (N\,\mathrm{grad}\,g)_\phi &= \frac{1}{2A_\phi}(g_i df(e_{jk}) + g_j df(e_{ki}) + g_k df(e_{ij})), \\ (N\,\mathrm{grad}\,g)_\phi \times df(e_{ij}) &= (g_j - g_i)N_\phi. \end{aligned}$$

So,

$$d\dot{f}(e_{ij}) = \frac{\nu_i + \nu_j}{2} df(e_{ij}) + \big(\cos\rho_\phi N_\phi (\mathrm{grad}\,v)_\phi - (\nu_j - \nu_i)\frac{\cot\beta_k}{2} N_\phi\big) \times df(e_{ij}).$$

On the other hand, since the deformation is conformal, there exists unique functions

$$\begin{aligned} &g : V \to \mathbb{R}, \\ &\omega : F \to \mathbb{R}, \\ &Y : F \to \mathrm{Im}\,\mathbb{H} \text{ and } Y_\phi \perp N_\phi \end{aligned}$$

such that on any oriented edge e_{ij}, we have

$$\dot{\lambda}_{e_{ij}} = \frac{g_i + g_j}{2} - \frac{\omega_\phi N_\phi}{2} - \frac{Y_\phi}{2} + \frac{\cot\beta_k}{2}(g_j - g_i)N_\varphi \times df(e_{ij})$$

and so

$$\begin{aligned} d\dot{f}(e_{ij}) &= \bar{\lambda}_{e_{ij}} df(e_{ij}) + df(e_{ij})\lambda_{e_{ij}} \\ &= (g_i + g_j) df(e_{ij}) + ((\omega_\phi - \cot\beta_k(g_j - g_i))N_\varphi + Y_\phi) \times df(e_{ij}). \end{aligned}$$

Comparing the two expression of $d\dot{f}(e_{ij})$, we have

$$\begin{aligned} g_i &= \frac{\nu_i}{2} \\ Y_\phi &= \cos\rho_\phi N_\phi \times \operatorname{grad} v_\phi \\ \omega_\phi &= 0. \end{aligned}$$

Notice that any conformal deformation is given by some $(u, Z) \in \mathbb{R}^{V+3F}$. Consider the normal deformation given by u on vertices as above, we get $(u, Y) \in \mathbb{R}^{V+3F}$. Then, the $\dot{\lambda}$ induced by $(0, Z - Y)$ is again an infinitesimal conformal deformation of the surface, which is even an infinitesimal rigid deformation, since it leaves edge lengths fixed. Here we recall a theorem which implies for convex triangulated surfaces the only rigid deformations are Euclidean transformations. A detailed discussion can be found in Pak (2010).

Theorem 4.3. *(Dehn's Theorem on Infinitesimal Rigidity of Convex Polytopes) Every simplicial convex polytope in* $\mathbb{R}^3$ *is infinitesimally rigid.*

Thus, as an analogue to the case of conformal deformations of the smooth sphere, we have

Theorem 4.4. *For a convex triangulated surface circumscribed in the unit sphere, infinitesimal normal deformations are the only infinitesimal conformal deformations up to infinitesimal rotations and translations.*

3. Double Inversions

In this section, infinitesimal deformations of smooth surfaces and triangulated surfaces under double inversions are compared. It demonstrates the nice relation of $\dot{\rho}$ between smooth and triangulated surfaces. The example in the smooth case is considered in Crane (2013).

Suppose we have 1-parameter family of inversions given $c_t \in \mathbb{R}^3$ with $c_0 = 0$ and $\dot{c}_0 =: a$. A mapping $f : M \to \mathbb{R}^3$ transforms under double inversion via

$$f \mapsto \overline{(\bar{f}^{-1} - c_t)}^{-1} = \frac{\frac{f}{||f||^2} - c_t}{||\frac{f}{||f||^2} - c_t||^2}.$$

We differentiate both sides with respect to t and set $t = 0$ and get

$$\begin{aligned} \dot{f} =& \frac{||\frac{f}{||f||^2} - c_t||^2(-a) - (\frac{f}{||f||^2} - c_t)2\langle -a, \frac{f}{||f||^2} - c_t\rangle}{||\frac{f}{||f||^2} - c_t||^4}\Big|_{t=0} \\ =& - a||f||^2 + 2f\langle a, f\rangle. \end{aligned}$$

We can write the result into products of $\mathbb{H}$-valued functions.

$$\begin{aligned} \dot{f} =& - a||f||^2 + 2f\langle a, f\rangle \\ =& a||f||^2 - 2a||f||^2 + 2f\langle a, f\rangle \\ =& - (af + 2f \times a)f \\ =& - faf. \end{aligned}$$

Assume the mapping $f : M \to \mathbb{R}^3$ is a smooth immersion. Then taking exterior derivative of both sides yields

$$d\dot{f} = -dfaf - fadf.$$

Since for a 1-parameter family of spin transformations satisfying $df_t = \bar{\lambda}_t df \lambda_t$ with $\lambda_0 = 1$, we have at $t = 0$ the formula

$$d\dot{f} = \dot{\bar{\lambda}}df + df\dot{\lambda}.$$

Comparing it with the previous equation, we get

$$\dot{\lambda} = -af.$$

We decompose a into tangential and normal components by denoting $a =: b + \langle a, N\rangle N$. Then, for any tangent vector X,

$$\begin{aligned}
-df \wedge d\dot{\lambda}(X, JX) =& df \wedge adf(X, JX) \\
=& df(X)aNdf(X) + df(X)Nadf(X) \\
=& df(X)(bN + Nb - 2\langle a, N\rangle)df(X) \\
=& 2\langle N, a\rangle |df(X)|^2.
\end{aligned}$$

Hence,

$$\dot{\rho} = \mathrm{D}\,\dot{\lambda} = 2\langle N, a\rangle.$$

We compare it with the deformation of triangulated surfaces under double inversions.

For the change of an edge,

$$\begin{aligned}
\dot{f}_j - \dot{f}_i =& -a(||f_j||^2 - ||f_i||^2) + 2(f_j\langle a, f_j\rangle - f_i\langle a, f_i\rangle) \\
=& \langle a, f_j + f_i\rangle(f_j - f_i) - a(||f_j||^2 - ||f_i||^2) + f_j\langle a, f_j\rangle - f_i\langle a, f_i\rangle + f_i\langle a, f_j\rangle \\
& - f_j\langle a, f_i\rangle \\
=& \langle a, f_j + f_i\rangle(f_j - f_i) + (f_i + f_j)\langle a, f_j - f_i\rangle - a\langle f_j - f_i, f_j + f_i\rangle \\
=& \langle a, f_j + f_i\rangle(f_j - f_i) + (f_j - f_i) \times \big((f_i + f_j) \times a\big).
\end{aligned}$$

Hence, the deformation is conformal with conformal factors on vertices $u_i := \langle a, f_i\rangle$. Defining on unoriented edges $W_{ij} = (f_i + f_j) \times a$, we have $(u, W^{\perp}) \in \mathbb{R}^{V+2E}$ and the corresponding quaternionic function on oriented edges are given by

$$\dot{\lambda}_{ij} = \frac{\langle a, f_i\rangle + \langle a, f_j\rangle}{2} + \frac{(f_i + f_j)}{2} \times a.$$

Then, we know $\mathrm{Im}(\mathrm{D}^*(u, W)) = 0$ since the deformation is locally closed. We calculate $\dot{\rho} = \mathrm{Re}(\mathrm{D}^*(u, W))$. We use the property that $\sum_{ij \in E:i} df(*e_{ij}) = 0$.

$$\begin{aligned}
\dot{\rho}_i =& \sum_{ij \in E:i} -\langle df(*e_{ij}), (f_i + f_j) \times a\rangle \\
=& \sum_{ij \in E:i} -\langle df(*e_{ij}), (f_j - f_i) \times a\rangle + 2 \sum_{ij \in E:i} \langle df(*e_{ij}), f_i \times a\rangle \\
=& \sum_{ij \in E:i} -\langle a, df(*e_{ij}) \times df(e_{ij})\rangle + 0 \\
=& \sum_{ij \in E:i} -\langle a, \big((\frac{\cot\beta_k}{2} N_{\phi_{ijk}} + \frac{\cot\beta_{\tilde{k}}}{2} N_{\phi_{i\tilde{k}j}}) \times df(e_{ij})\big) \times df(e_{ij})\rangle \\
=& \sum_{ij \in E:i} \langle a, (\frac{\cot\beta_k}{2} N_{\phi_{ijk}} + \frac{\cot\beta_{\tilde{k}}}{2} N_{\phi_{i\tilde{k}j}})|df(e_{ij})|^2\rangle \\
=& \sum_{ijk \in F:i} \langle a, (\frac{\cot\beta_k}{2}|df(e_{ij})|^2 + \frac{\cot\beta_j}{2}|df(e_{ik})|^2) N_{\phi_{ijk}}\rangle \\
=& 2 \sum_{ijk \in F:i} A_{ijk,i}\langle a, N_{\phi_{ijk}}\rangle
\end{aligned}$$

where $A_{ijk,i} = \frac{\cot\beta_k}{4}|df(e_{ij})|^2 + \frac{\cot\beta_j}{4}|df(e_{ik})|^2$ is the shaded area of the triangle (ijk) shown in Figure 4.1.

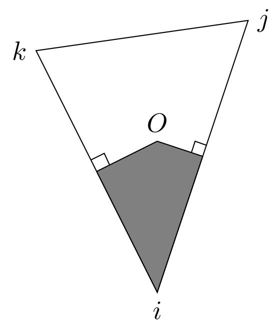

FIGURE 4.1. $A_{ijk,i}$ is the area of the shaded portion and O is the circumcenter.

Theorem 4.5. *Consider the infinitesimal double inversions generated by*

$$f \mapsto \overline{(\bar{f}^{-1} - c_t)}^{-1} = \frac{\frac{f}{||f||^2} - c_t}{||\frac{f}{||f||^2} - c_t||^2}.$$

where $c_0 = 0, \dot{c} = a$. *Then, for an immersed smooth surface* $f : M \to \mathbb{R}^3$, *we have*

$$\dot{\rho} = \mathrm{D}\,\dot{\lambda} = 2\langle N, a\rangle$$

where $\rho|df|$ *is the change of mean curvature half density.*

On the other hand, for an immersed triangulated surface $f : M \to \mathbb{R}^3$, *we have*

$$\dot{\rho} = 2 \sum_{ijk\in F:i} A_{ijk,i}\langle a, N_{\phi_{ijk}}\rangle.$$

4. Examples about the Kernel of the Discrete Dirac Operator

We further look at deformations of the unit sphere in order to find examples of surfaces with $\dim(\mathrm{Ker}\,\mathrm{D}) = 4$ and $\dim(\mathrm{Ker}\,\mathrm{D}) > 4$.

We firstly consider the case $\dot{\rho} \equiv 0$ under a normal deformation. As in section 2, a normal deformation is determined by $\nu : V \to \mathbb{R}$ and we have the corresponding element $(g, Z) \in \mathbb{R}^{V+3F}$, where

$$\begin{aligned} g_i =& \frac{\nu_i}{2}, \\ Z_{ijk} :=& -(\frac{\omega_{ijk} N_{ijk}}{2} + \frac{Y_{ijk}}{2}), \\ =& -\frac{1}{2}\cos\rho_{ijk} N_{ijk} \times (grad\nu), \\ =& -\frac{1}{2}\cos\rho_{ijk}\frac{1}{2A_{ijk}}\big(\nu_i df(e_{jk}) + \nu_j df(e_{ki}) + \nu_k df(e_{ij})\big). \end{aligned}$$

We can then calculate $\dot{\rho}$ under the "normal deformation"'.

$$\begin{aligned} \dot{\rho}_i :=& \mathrm{Re}\big(\mathrm{D}(g, Z)\big) \\ =& \sum_{ij\in E:i} \langle df(e_{ij}), dZ(*e_{ij})\rangle \\ =& \frac{1}{2}\sum_{ij\in E:i} \langle df(e_{ij}), \frac{\cos\rho_{ijk}}{2A_{ijk}}\big(\nu_i df(e_{jk}) + \nu_j df(e_{ki}) + \nu_k df(e_{ij})\big) - \\ & \frac{\cos\rho_{ij\tilde{k}}}{2A_{ij\tilde{k}}}\big(\nu_i df(e_{j\tilde{k}}) + \nu_j df(e_{\tilde{k}i}) + \nu_{\tilde{k}} df(e_{ij})\big)\rangle \end{aligned}$$

$$
\begin{aligned}
&= \sum_{ijk\in F:i} \frac{\cos\rho_{ijk}}{4A_{ijk}} \langle df(e_{ij}) - df(e_{ik}), \nu_i df(e_{jk}) + \nu_j df(e_{ki}) + \nu_k df(e_{ij})\rangle \\
&= \sum_{ijk\in F:i} \frac{\cos\rho_{ijk}}{4A_{ijk}} \langle df(e_{kj}), \nu_i df(e_{jk}) + \nu_j df(e_{ki}) + \nu_k df(e_{ij})\rangle \\
&= \sum_{ijk\in F:i} \frac{\cos\rho_{ijk}}{2} (-\nu_i \frac{|df(e_{jk})|^2}{2A_{ijk}} + \nu_j \cot\beta_k + \nu_k \cot\beta_j) \\
&= \sum_{ijk\in F:i} \frac{\cos\rho_{ijk}}{2} \big(-\nu_i \frac{|df(e_{jk})|(|df(e_ij)|\cos\beta_j + |df(e_ik)|\cos\beta_k)}{2A_{ijk}} \\
&\qquad + \nu_j \cot\beta_k + \nu_k \cot\beta_j \big) \\
&= \sum_{ijk\in F:i} \frac{\cos\rho_{ijk}}{2} ((\nu_j - \nu_i)\cot\beta_k + (\nu_k - \nu_i)\cot\beta_j) \\
&= \sum_{ij\in E;i} (\nu_j - \nu_i)(\frac{\cos\rho_{ijk}\cot\beta_k}{2} + \frac{\cos\rho_{ij\tilde{k}}\cot\beta_{\tilde{k}}}{2}).
\end{aligned}
$$

Theorem 4.6. *Suppose the coefficients* $\frac{\cos\rho_{ijk}\cot\beta_k}{2} + \frac{\cos\rho_{ij\tilde{k}}\cot\beta_{\tilde{k}}}{2}$ *are positive on every edge. Then the only normal deformation with* $\dot{\rho} \equiv 0$ *is constant scaling, i.e.* $\nu \equiv$ *constant.*

PROOF. Suppose ν is not constant, then exist some vertex $i \in V$ such that

$$\nu_i \geq \nu_j \quad \forall j \in V \text{ and } ij \in E$$

and the inequality is strict for at least one neighboring vertex. Hence,

$$
\begin{aligned}
\dot{\rho}_i &= \sum_{ij} (\nu_i - \nu_j)(\frac{\cos\rho_{ijk}\cot\beta_k}{2} + \frac{\cos\rho_{ij\tilde{k}}\cot\beta_{\tilde{k}}}{2}) \\
&> 0
\end{aligned}
$$

which contradicts that $\dot{\rho}$ vanishes identically. Hence ν is constant. □

Now we calculate $\dot{\rho}$ under a rigid deformation.

Theorem 4.7. *Given a triangulated surface circumscribed in the unit sphere* $f : M \to \mathbb{S}^2$. *Suppose a rigid deformation is given by* $(0, a_{ij}\, T_{ij} \times \frac{f_i+f_j}{2} + h_{ij}\frac{f_i+f_j}{2}) \in \mathbb{R}^{V+2E}$*.Then,* $\dot{\rho}$ *vanishes identically, i.e. for all vertices i,*

$$\dot{\rho}_i = \mathrm{Re}\left(\mathrm{D}^*(0, a_{ij}T_{ij} \times \frac{f_i+f_j}{2} + h_{ij}\frac{f_i+f_j}{2})\right)_i = 0.$$

PROOF. Suppose a rigid deformation is given by $(0, a_{ij}\, T_{ij} \times \frac{f_i+f_j}{2} + h_{ij}\frac{f_i+f_j}{2}) \in \mathbb{R}^{V+2E}$. Notice that

$$a_{ij}T_{ij} \times \frac{f_i+f_j}{2} + h_{ij}\frac{f_i+f_j}{2} = a_{ji}T_{ji} \times \frac{f_i+f_j}{2} + h_{ji}\frac{f_i+f_j}{2}.$$

Thus,

$$a_{ij} = -a_{ji} \quad , \quad h_{ij} = h_{ji}.$$

Also, on a face $\phi = (ijk)$,

$$
\begin{aligned}
&(\mathrm{Im}\, \mathrm{D}^*(0, h_{ij}\frac{f_i+f_j}{2}))_\phi \\
&= -\, h_{ij}(df(e_{ij}) \times f_\phi - \frac{\cot\beta_k}{2}|df(e_{ij})|^2 N_\phi) \\
&\quad -\, h_{jk}(df(e_{jk}) \times f_\phi - \frac{\cot\beta_i}{2}|df(e_{jk})|^2 N_\phi)
\end{aligned}
$$

$$
\begin{aligned}
&- h_{ki}(df(e_{ki}) \times f_\phi - \frac{\cot\beta_j}{2}|df(e_{ki})|^2 N_\phi)\\
=&\big(h_{ij}\frac{\cot\beta_k}{2}|df(e_{ij})|^2 + h_{jk}\frac{\cot\beta_i}{2}|df(e_{jk})|^2 + h_{ki}\frac{\cot\beta_j}{2}|df(e_{ki})|^2\big)N_\phi\\
&- \cos\rho_\phi\big(h_{ij}df(e_{ij}) + h_{jk}df(e_{jk}) + h_{ki}df(e_{ki})\big) \times N_\phi
\end{aligned}
$$

and

$$
\begin{aligned}
&\operatorname{Im}\mathrm{D}^*(0, a_{ij}T_{ij} \times \frac{f_i+f_j}{2})_\phi\\
=&-(-a_{ij}|df(e_{ij})|\frac{f_i+f_j}{2} - a_{jk}|df(e_{jk})|\frac{f_j+f_k}{2} - a_{ki}|df(e_{ki})|\frac{f_k+f_i}{2})\\
=&(a_{ij}|df(e_{ij})| + a_{jk}|df(e_{jk})| + a_{ki}|df(e_{ki})|)\cos\rho_\phi N_\phi\\
&+ a_{ij}|df(e_{ij})|\frac{\cot\beta_k}{2}df(e_{ij}) \times N_\phi\\
&+ a_{jk}|df(e_{jk})|\frac{\cot\beta_i}{2}df(e_{jk}) \times N_\phi\\
&+ a_{ki}|df(e_{ki})|\frac{\cot\beta_j}{2}df(e_{ki}) \times N_\phi.
\end{aligned}
$$

Since $(0, a_{ij}\,T_{ij} \times \frac{f_i+f_j}{2} + h_{ij}\frac{f_i+f_j}{2})$ gives an infinitesimal conformal deformation, we have

$$
\begin{aligned}
0 =& \operatorname{Im}\mathrm{D}^*(0, a_{ij}T_{ij} \times \frac{f_i+f_j}{2} + h_{ij}\frac{f_i+f_j}{2}) \qquad (4.1)\\
=& \big(a_{ij}|df(e_{ij})|\frac{\cot\beta_k}{2} - h_{ij}\cos\rho_\phi\big)df(e_{ij}) \times N_\phi\\
&+ \big(a_{jk}|df(e_{jk})|\frac{\cot\beta_i}{2} - h_{jk}\cos\rho_\phi\big)df(e_{jk}) \times N_\phi\\
&+ \big(a_{ki}|df(e_{ki})|\frac{\cot\beta_j}{2} - h_{ki}\cos\rho_\phi\big)df(e_{ki}) \times N_\phi\\
&+ \big((a_{ij}|df(e_{ij})| + a_{jk}|df(e_{jk})| + a_{ki}|df(e_{ki})|)\cos\rho_\phi\\
&+ h_{ij}\frac{\cot\beta_k}{2}|df(e_{ij})|^2 + h_{jk}\frac{\cot\beta_i}{2}|df(e_{jk})|^2 + h_{ki}\frac{\cot\beta_j}{2}|df(e_{ki})|^2\big)N_\phi.
\end{aligned}
$$

Since $df(e_{ij})$,$df(e_{jk})$ and $df(e_{ki})$ span an affine plane and $df(e_{ij})+df(e_{jk})+df(e_{ki}) = 0$, for each face ϕ, there exists $\omega_\phi \in \mathbb{R}$ such that

$$
\begin{aligned}
\omega_\phi =& a_{ij}|df(e_{ij})|\frac{\cot\beta_k}{2} - h_{ij}\cos\rho_\phi\\
=& a_{jk}|df(e_{jk})|\frac{\cot\beta_i}{2} - h_{jk}\cos\rho_\phi\\
=& a_{ki}|df(e_{ki})|\frac{\cot\beta_j}{2} - h_{ki}\cos\rho_\phi.
\end{aligned}
$$

We now then calculate the real part of D^*.

$$
\begin{aligned}
&\operatorname{Re}(\mathrm{D}^*(0, a_{ij}T_{ij} \times \frac{f_i+f_j}{2} + h_{ij}\frac{f_i+f_j}{2}))_i\\
=&-\sum_{ij\in E:i}\langle a_{ij}T_{ij} \times \frac{f_i+f_j}{2} + h_{ij}\frac{f_i+f_j}{2}, df(*e_{ij})\rangle\\
=&\sum_{ij\in E:i}\langle a_{ij}T_{ij} \times \frac{f_i+f_j}{2} + h_{ij}\frac{f_i+f_j}{2}, \frac{\cot\beta_k}{2}df(e_{ij}) \times N_\phi + \frac{\cot\beta_{\tilde{k}}}{2}df(e_{ij}) \times N_\phi)\rangle\\
=&\sum_{ij\in E:i}\langle a_{ij}\frac{f_i+f_j}{2}, \frac{\cot\beta_k}{2}|df(e_{ij})|N_\phi + \frac{\cot\beta_{\tilde{k}}}{2}|df(e_{ij})|N_{\tilde{\phi}})\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+ \langle h_{ij}\frac{f_i+f_j}{2}, \frac{\cot\beta_k}{2} df(e_{ij}) \times N_\phi + \frac{\cot\beta_{\tilde{k}}}{2} df(e_{ij}) \times N_{\tilde{\phi}})\rangle \\
=& \sum_{ij\in E:i} a_{ij}|df(e_{ij})|(\frac{\cot\beta_k}{2}\cos\rho_{ijk} + \frac{\cot\beta_{\tilde{k}}}{2}\cos\rho_{ij\tilde{k}}) \\
&\quad + h_{ij}|df(e_{ij})|^2((\frac{\cot\beta_k}{2})^2 - (\frac{\cot\beta_{\tilde{k}}}{2})^2) \\
=& \sum_{ij\in E:i} a_{ij}|df(e_{ij})|\frac{\cot\beta_k}{2}\cos\rho_{ijk} - a_{ji}|df(e_{ij})|\frac{\cot\beta_{\tilde{k}}}{2}\cos\rho_{ij\tilde{k}} \\
&\quad + h_{ij}|df(e_{ij})|^2((\frac{\cot\beta_k}{2})^2 - (\frac{\cot\beta_{\tilde{k}}}{2})^2) \\
=& \sum_{ij\in E:i} (\omega_{ijk}\cos\rho_{ijk} + h_{ij}\cos^2\rho_{ijk}) - (\omega_{ij\tilde{k}}\cos\rho_{ij\tilde{k}} + h_{ij}\cos^2\rho_{ij\tilde{k}}) \\
&\quad + h_{ij}|df(e_{ij})|^2((\frac{\cot\beta_k}{2})^2 - (\frac{\cot\beta_{\tilde{k}}}{2})^2) \\
=& \sum_{ij\in E:i} (\omega_{ijk}\cos\rho_{ijk} + h_{ij}|\frac{f_i+f_j}{2}|^2) - (\omega_{ij\tilde{k}}\cos\rho_{ij\tilde{k}} + h_{ij}|\frac{f_i+f_j}{2}|^2) \\
=&0.
\end{aligned}
$$

In particular from theorem 3.26, we have on all vertices

$$
\begin{aligned}
\sum_{ij\in E:i} \frac{\dot{\alpha}_{ij}}{2}|df(e_{ij})| = \dot{\rho}_i =& \operatorname{Re}\left(\mathrm{D}^*(0, a_{ij}T_{ij} \times \frac{f_i+f_j}{2} + h_{ij}\frac{f_i+f_j}{2})\right)_i \\
=&0.
\end{aligned}
$$

□

Example 2. *A regular octahedron can be circumscribed in a sphere and is convex. By Theorem 4.4, normal deformations are the only conformal deformations. And constant scaling is the only normal deformations having $\dot{\rho}$ identically zero by Theorem 4.6. Thus, it has* $\dim(\operatorname{Ker}\mathrm{D}) = 4$.

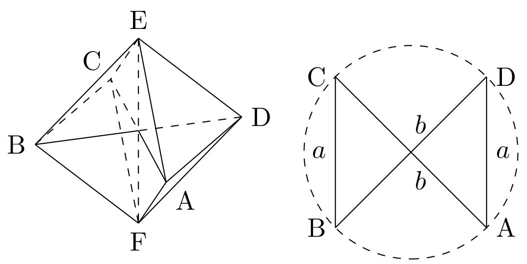

FIGURE 4.2. Biscard's octahedron and its top view

Example 3. *Bricard's octahedron is a flexible almost immersed octahedron which can be circumscribed in a sphere. Its infinitesimal rigid deformation gives $\dot{\rho}$ identically zero by Theorem 4.7. Hence,* $\dim(\operatorname{Ker}\mathrm{D}) > 4$.

Remark 4.8. *In general, $\dot{\rho}$ is not identically zero for rigid deformations. The following flexible torus is discussed in Pak (2010), which gives a counterexample here.*

Example 4. *Consider an immersed torus, consisting of* 4 *identical rhombi and* 4 *identical tubes connecting the rhombi as in Figure 4.3.*

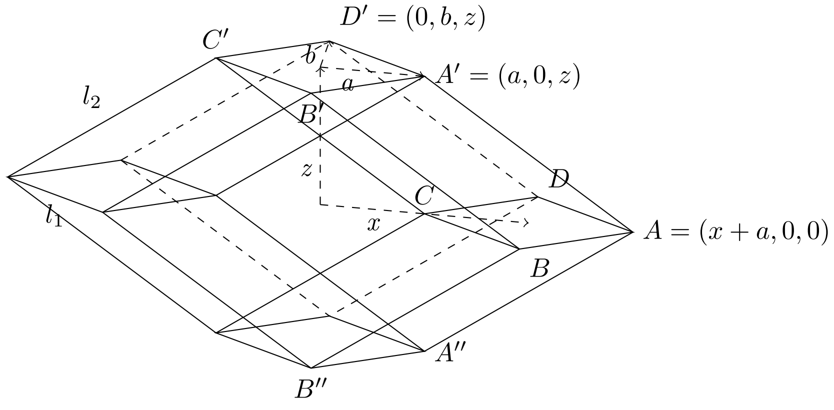

FIGURE 4.3. Flexible Torus

We claim that the torus is flexible. By symmetry, we can focus on the deformations of the tube $ABCDA'B'C'D'$ and have

$$\cos\angle AA'B' = \frac{\langle(-a,-b,0),(x,0,z)\rangle}{l_1 l_2} = \frac{-ax}{l_1 l_2},$$
$$\cos\angle BB'C' = \frac{\langle(-a,b,0),(x,0,z)\rangle}{l_1 l_2} = \frac{-ax}{l_1 l_2}.$$

In addition with symmetries, we get

$$\beta := \angle AA'B' = \angle BB'C' = \angle CC'D' = \angle DD'D'$$

and so the four parallelograms are isometric. The configuration of the torus is given by 4 parameters: x, z, a, b. And we have three constraints under isometric deformations,

$$a^2 + b^2 = {l_1}^2,$$
$$x^2 + z^2 = {l_2}^2,$$
$$\frac{-ax}{l_1 l_2} = \cos\beta,$$

where the right hand sides are constants. We can then express z, a, b in terms of a free variable x. Hence the torus is flexible. By adding diagonals AB', CB', AD', CD' on each tube, we get a flexible triangulated torus.

We now show that $\dot\rho$ is not identically zero under some rigid deformation. Suppose $x = x(t)$. We have

$$N_{ABA'B'} = \frac{(-a,-b,0)\times(x,0,-z)}{l_1 l_2 \sin\beta} = \frac{(bz,-az,bx)}{l_1 l_2 \sin\beta},$$
$$N_{ADA'D'} = \frac{(a,-b,0)\times(x,0,-z)}{l_1 l_2 \sin\beta} = \frac{(bz,az,bx)}{l_1 l_2 \sin\beta},$$
$$N_{ABA''B''} = \frac{(x,0,z)\times(-a,-b,0)}{l_1 l_2 \sin\beta} = \frac{(bz,-az,-bx)}{l_1 l_2 \sin\beta},$$
$$N_{ADA''D''} = \frac{(-a,b,0)\times(x,0,z)}{l_1 l_2 \sin\beta} = \frac{(bz,az,-bx)}{l_1 l_2 \sin\beta}.$$

Notice that

$$(l_1 l_2 \sin\beta)^2 = b^2 z^2 + a^2 z^2 + b^2 x^2.$$

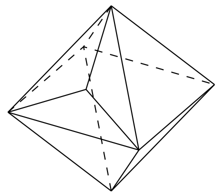

FIGURE 4.4. A regular octahedron with a new vertex

We then have

$$\begin{aligned}
\cos\alpha_{AA'} &= \langle N_{ADA'D'}, N_{ABA'B'}\rangle = 1 - \frac{2a^2z^2}{(l_1l_2\sin\beta)^2},\\
\sin\alpha_{AA'} &= |N_{ADA'D'} \times N_{ABA'B'}| = \frac{2abzl_2}{(l_1l_2\sin\beta)^2},\\
\cos\alpha_{AB} &= \langle N_{ABA'B'}, N_{ABA''B''}\rangle = 1 - \frac{2b^2x^2}{(l_1l_2\sin\beta)^2},\\
\sin\alpha_{AB} &= |N_{ABA'B'} \times N_{ABA''B''}| = \frac{2bxzl_1}{(l_1l_2\sin\beta)^2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\dot\alpha_{AA'} &= \frac{(\cos\dot{\alpha}_{AA'})}{\sin\alpha_{AA'}} = \frac{-2(\dot az + a\dot z)}{bl_2},\\
\dot\alpha_{AB} &= \frac{(\cos\dot{\alpha}_{AB})}{\sin\alpha_{AB}} = \frac{-2(\dot bx + b\dot x)}{zl_1}.
\end{aligned}$$

Differentiating the constants, we have identities

$$\begin{aligned}
\dot aa + \dot bb &= 0,\\
\dot xx + \dot zz &= 0,\\
\dot ax + \dot xa &= 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
\dot\rho_A =& \frac{\dot\alpha_{AB}l_1}{2} + \frac{\dot\alpha_{AA'}l_2}{2} + \frac{\dot\alpha_{AD}l_1}{2} + \frac{\dot\alpha_{AA''}l_2}{2}\\
=& -2(\frac{\dot bx + b\dot x}{z} + \frac{\dot az + a\dot z}{b})\\
=& \frac{-2\dot x}{bzx}({xl_1}^2 - {al_2}^2),
\end{aligned}$$

which is not zero generally.

One may wonder if there is any relation between infinitesimal rigidity and dim(Ker D).

Example 5. *Consider choosing a triangle of a regular octahedron and adding new edges connecting its circumcenter with the three vertices of the triangle. This gives a new triangulated surface as shown in Figure 4.4, which is continuously rigid but not infinitesimal rigid. This surface has* dim(Ker D) $= 4$*, which is explained as follows:*

The new vertex can only moves orthogonal to the triangle. $\dot\rho > 0$ *if the vertex moving outward and* $\dot\rho < 0$ *if the vertex moving inward. So to have* $\dot\rho = 0$ *on the vertex, it has to stay on the triangle. Hence, to consider the conformal deformation with* $\dot\rho \equiv 0$*, one can ignore the new vertex. Thus, the dimension of* Ker D *is same as without the new vertex.*

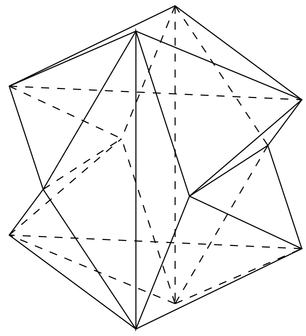

FIGURE 4.5. Jessen's orthogonal icosahedron

Example 6. *Jessen's orthogonal icosahedron is obtained from a regular icosahedron by flipping* 6 *edges symmetrically (Goldberg, 1978; Jessen, 1967). Its vertices are the same as the regular icosahedron and hence can be circumscribed in a sphere. It is continuously rigid but not infinitesimal rigid. Its infinitesimal rigid deformation gives $\dot{\rho}$ identically zero by Theorem 4.7. Hence,* $\dim(\operatorname{Ker} D) > 4$.

APPENDIX A

Zusammenfassung

Diese Arbeit behandelt eine diskretisierte Version von Quaternischer Analysis. Wir beschreiben die Beziehung zwischen infinitesimalem konformen Verformungen der triangulierten Flächen im euklidischen Raum und ihrer extrinsische Geometrie. Die Arbeit folgt dem Ziel der diskreten Differentialgeometrie (Bobenko and Suris, 2008), um mathematische Strukturen von triangulierten Flächen ähnlich wie in der glatten Theorie zu suchen.

Diese Arbeit ist in 4 Kapitel unterteilt. Kapitel 1 enthält Hintergrundwissen über quaternionische lineare Algebra und diskrete Differentialformen. Die Hodge Zerlegung für diskrete Differentialformen wird hergeleitet. Dieses Kapitel endet mit einer eindimensionalen Darstellung der infinitesimalem konformen Verformungen von Oberflächen. Wir vergleichen den Raum der ebenen Kurven mit fester Länge, im glatten wie im diskreten Fall parametrisiert durch Krümmungfunktionen.

Kapitel 2 behandelt konforme Deformationen von glatten Oberflächen mithilfe quaternionischer Analysis. Wir fassen die grundlegenden Ergebnisse zusammen und konzentrieren uns auf infinitesimale konforme Verformungen. Bedingungen für infinitesimale konforme Deformationen für Oberflächen von höherem Geschlecht abgeleitet werden. Die meisten Sätze haben hier ein diskretes Analogen im folgenden Kapitel.

Kapitel 3 ist der wichtigste Teil der Arbeit und konzentriert sich auf infinitesimale konforme Deformationen von triangulierten Flächen. Wir untersuchen den Begriff konformer Äquivalenz von triangulierte Flächen und seine Eigenschaften im Vergleich zu der glatten Theorie. Dann leiten wir den diskreten Dirac-Operator. Bedingungen für infinitesimale konforme Deformationen für Oberflächen von höherem Geschlecht werden gezeigt. Es endet mit der Herleitung des diskreten Laplace-Operators bzw. der Kotangens Laplace Formel.

Kapitel 4 gibt explizite Beispiele zum Vergleich. Die Wahl des Begriffs der konformen Äquivalenz und diskreten Definition in Kapitel 3 wird durch den Vergleich von glatten und diskretes Theorie gerechtfertigt. Die Dimension des Kerns des diskreten Dirac Operators wird auch für einige Triangulierte Flächen berechnet. Wir präsentieren Beispiele mit $\dim(\operatorname{Ker} \mathrm{D}) = 4$ und $\dim(\operatorname{Ker} \mathrm{D}) > 4$. Sie werden aus dem Studium der Steifigkeit der polyedrischen Flächen motiviert.

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